Volatility information with Levy Processes

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1 Outline

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- Conclusion

2 Introduction

2.1 Levy processes in finance

In standard Option pricing theory: driving process for modeling risk factors and stochasticity based on Brownian motion (Bachelier (1900), Samuelson (1965)).

Empirical literature (MandelBrot (1960) investigated model driven by more general processes: α-stable Levy motion. Madan & Seneta (1987, 1990) suggested another Levy process: variance gamma process.

More generally, Barndoff-Nielsen (1977) introduced generalized inverse Gaussian distributions for sandpile. Eberlein (1995) suggested to model risk factors S_t as the exponential of a generalized hyperbolic Levy process X_t :

$$S_t = S_0 \exp\left(X_t\right). \tag{2.1}$$

Special cases are the hyperbolic model (Eberlein and Keller (1995)) and the normal inverse Gaussian model (Barndoff-Nielsen (1998)).

Subject of the talk: How to get information from option prices to infer the driving Levy process? How to model "volatility surface"? How to finally price options in a suitable way for calibration?

2.2 Problem/Difficulties of using Levy processes:

- Measure?

Using general results from semimartingale theory, the no-arbitrage condition leads to the existence of a class of equivalent martingale measures. In an incomplete markets, this set is infinite. Hence, how to decide which measure to use? Substantial work of Terence Chan (1999) using Föllmer-Schweizer minimal measure and a minimization of the relative entropy with respect to the canonical measure (result based on the Esscher transform).

- Integro-differential equation:

Derivation of the Integro-differential Equation:

$$dX_t = b\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dW_t + J\left(t, X_t\right) dN_t \tag{2.2}$$

Levy integro-differential equation:

$$\begin{aligned} & \frac{\partial}{\partial t} P\left(x, t\right) + \frac{\partial}{\partial x} P\left(x, t\right) b\left(t, x\right) + \frac{1}{2} \sigma^2\left(t, X_t\right) \frac{\partial^2}{\partial x^2} P\left(x, t\right) \\ & + \int_{J \in E} \left[P\left(x+y, t\right) - P\left(x, t\right)\right] .p\left(dJ, dN_t\right) \\ = \; & rP\left(x, t\right) \end{aligned} \tag{2.3}$$

with the boundary

$$P\left(x, T\right) = Payoff\left(e^x\right) \tag{2.4}$$

- Related literature/Works:

Writing the process in the form of $X_t = \mu t + \sigma_t L_t$ for a standardized Levy process $\left(L_t\right)_{t \geq 0}$, Barndoff and Shepard (2001) showed how to account for stochastic volatility.

Matacz (1997), Boyarchencko Levendorskii (1999) used truncated Levy pro-

cesses to price options. Rydberg (1999), Kiesel Bingham(1999) used generalized hyperbolic diffusion to price options.

Björk, di Masi Kabanov and Runggaldier (1997) showed how to use an appropriate Levy process to price zero coupon bonds consistently with the Heath-Jarrow-Morton conditions. Raible (2000) showed the unicity of the martingale measure for this model.

Terence Chan (1999) using Föllmer-Schweizer minimal measure and a minimization of the relative entropy with respect to the canonical measure (result based on the Esscher transform) showed how to derive an appropriate equivalent martingale measure.

3 Volatility smile for Brownian diffusion: Dupire(1994)

3.1 Non-parametric calibration

Assumption: one factor model driven by a Brownian diffusion in the risk neutral probability measure

$$dS_t = rS_t dt + \sigma(t, S_t) S_t dW_t \tag{3.1}$$

for simplicity assume $r = 0$

Breeden-Litzenberger(1978): call options payoff for all strikes K are an L^2 basis.

A call option being the financial right but not the obligation to buy a certain asset at the strike price K has the following payoff:

$$Payoff_Call(S, K, T) = (S_T - K)^+$$

Using distribution theory

$$\frac{\partial^2}{\partial K^2} Payoff_Call(S, K, t) = 1_{\{S_T = K\}}$$

Taking the expectation leads to

$$p\left(t, S_t, T, K\right) = \frac{\partial^2}{\partial K^2} Call\left(S, K, T\right), \tag{3.2}$$

where $p\left(t, S_t, T, K\right)$ is the transition probability from the state $\left(t, S_t\right)$ to $\left(T, K\right)$. But the transition probability needs to satisfy the Kolmogorov forward (or Fokker-Plank) equation

$$\frac{\partial}{\partial T} p\left(t, s, T, K\right) - \frac{1}{2}\frac{\partial^2}{\partial K^2}\left(\sigma^2\left(K, T\right) p\left(t, s, T, K\right)\right) = 0$$

or re-expressing the probability density with respect to options prices

$$\frac{\partial}{\partial T}\frac{\partial^2}{\partial K^2} Call\left(S, K, T\right) - \frac{1}{2}\frac{\partial^2}{\partial K^2}\left(\sigma^2\left(K, T\right)\frac{\partial^2}{\partial K^2} Call\left(S, K, T\right)\right) = 0 \tag{3.3}$$

Integrating twice and using standard limit leads to

$$\frac{\partial}{\partial T} Call\left(S, K, T\right) - \frac{1}{2}\sigma^2\left(K, T\right)\frac{\partial^2}{\partial K^2} Call\left(S, K, T\right) = 0 \tag{3.4}$$

or

$$\sigma^2\left(K, T\right) = \frac{2\frac{\partial}{\partial T} Call\left(S, K, T\right)}{\frac{\partial^2}{\partial K^2} Call\left(S, K, T\right)} \tag{3.5}$$

Strong result: from option prices information, not only do we get the final distribution

(2) but we do get the whole distribution. Practically, not easy to interpolate call options prices... problem of volatility surface for long dated options. forward smile surface...

3.2 Adaptation to Levy processes

No unicity of the measure. Need to assume that we can find a measure under which the driving process is an exponential Levy process.

The Kolmogorov forward equation changed into a more complexed integro-differential equation:

$$\frac{\partial}{\partial T}p\left(t,s,T,K\right)-\frac{1}{2}\frac{\partial^2}{\partial K^2}\left(\sigma^2\left(K,T\right)p\left(t,s,T,K\right)\right)+\mathbb{E}\left[p\left(t,s,T,K\right)-p\left(t,s,T,K\right)\right]=0 \tag{3.6}$$

The problem has become not easy at all.

Solution: use another method:

4 Levy processes

We assume that there exists a measure under which the driving factor $(S_t)_{t\in\mathbb{R}^+}$ can be modelled as a continuous time process, function of a geometric Brownian motion times the exponential of a Levy process with no Brownian part X_t

$$S_t = S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_T+X_t} \tag{4.7}$$

4.1 Some properties

Characterization of the Levy process by two means

- Laplace transform.
- Fourier transform.

4.1.1 Laplace transform

We assume that the Laplace transform of the Levy process is non exploding for $\lambda \leq \lambda_u$. Then, its Laplace transform must have the following form: there exists a function

$\phi :]-\infty, \lambda_u] \to \mathbb{R}$ defined as for every $t \in \mathbb{R}^+$, $\lambda \in]-\infty, \lambda_u]$

$$E\left[e^{\lambda X_t}\right] = e^{t\phi(\lambda)}$$

This function is called the Levy-Laplace exponent.

Moment of the Levy processes are easily derived as

$$\mathbb{E}\left[S_t^\lambda\right] = S_0^\lambda e^{t\left(\lambda\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2\lambda^2}{2}+\phi(\lambda)\right)} \tag{4.8}$$

Example 1 *Jump driven by Poisson process*

Jumps independent, identically distributed with distribution U

$$S_t = S_0 e^{\left(\mu-\sigma^2/2\right)t+\sigma W_t}\prod_{i=1}^{N_t}\left(1+U_j\right)$$

Poisson process with intensity θ

$$\mathbb{E}\left[S_t^\lambda\right] = S_0^\lambda e^{t\left(\lambda\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2\lambda^2}{2}+\theta\left(\mathbb{E}[1+U]^\lambda-1\right)\right)}$$

Levy exponent is

$$\phi\left(\lambda\right) = \theta\left(\mathbb{E}\left[1+U\right]^\lambda - 1\right) \tag{4.9}$$

$$\phi\left(1\right) = \theta\mathbb{E}\left[U\right]$$

4.1.2 Fourier transform

No restriction on the underlying Levy process. Levy-Khintchine exponent: there exists a function $\psi :]-\infty, +\infty[\rightarrow \mathbb{R}$ so that for every $t \in \mathbb{R}^+$, for every $\lambda \in \mathbb{R}$

$$\mathbb{E}\left[e^{i\lambda \ln S_t}\right] = S_0^{i\lambda} e^{t\psi(\lambda)}$$

with $\psi(\lambda) = i\mu\lambda - \frac{\sigma^2}{2}\lambda^2 - \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}} \Pi(dx)\right)$ with $\mu \in \mathbb{R}$ and Π, called the Levy measure, is a positive measure on $\mathbb{R}$ so that $\int_{\mathbb{R}} Min\left(1, x^2\right) \Pi(dx) < \infty$. The term $\frac{\sigma^2}{2}\lambda^2$ is the Brownian part of the process while the parameter μ is the drift. $\int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}} \Pi(dx)\right)$ is the term due to the pure "Levy" process.

Example 2 *Jump driven by Poisson process*

$$\mathbb{E}\left[e^{i\lambda \ln S_t}\right] = S_0^{i\lambda} e^{t\left(i\lambda\left(\mu - \frac{\sigma^2}{2}\right) - \frac{\sigma^2\lambda^2}{2} + \theta\left(\mathbb{E}[1+U]^{i\lambda} - 1\right)\right)} \tag{4.10}$$

Levy exponent is

$$\psi(\lambda) = i\lambda\left(\mu - \frac{\sigma^2}{2}\right) - \frac{\sigma^2\lambda^2}{2} + \theta\left(\mathbb{E}[1+U]^{i\lambda} - 1\right) \tag{4.11}$$

4.2 Option pricing

4.2.1 No arbitrage condition

$\left(e^{-rT}S_T\right)$ be a martingale under $\mathbb{P}$ leads to

$$S_0 e^{t\left(-r+\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2}{2}+\phi(1)\right)} = S_0$$

which means

$$\mu = r - \phi\left(1\right) \tag{4.12}$$

4.2.2 Pricing with Laplace transform

Theorem 1 *Similarly to the Hull and White approach for stochastic volatility (1987), option price can be rewritten as the weighted sum of Black Scholes prices:*

$$P^i = \mathbb{E}\left[P^i_{BS}\left(S_0 e^{-T\phi(1)}e^{X_T}, K, T, r, \sigma\right)\right] \tag{4.13}$$

where i stands for either call or put, $P^i_{BS}\left(S_0 e^{-T\phi(1)}e^{X_T}, K, T, r, \sigma\right)$ is the Black Scholes price with initial underlying price $S_0 e^{-T\phi(1)}e^{X_T}$, strike K, maturity T, risk free interest rate r, and Brownian volatility σ.

Proof: put option on S_T with strike can be seen as a call option on $(-S_T)$ with a strike $-K$ $(K - S_T)^+ = (-S_T - (-K))^+$
We only examine the case of the call option. The price of the option is calculated as the expectation of the discounted payoff

$$\begin{aligned}
&\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] \\
&= \mathbb{E}\left[e^{-rT}\left(S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_t} e^{X_T} - K\right)^+\right] \\
&= \mathbb{E}\left[e^{-rT}\left(\left(S_0 e^{-T\phi(1)}\right) e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_t} e^{X_T} - K\right)^+\right]
\end{aligned}$$

where we have used the no-arbitrage condition on the drift term (12). Using conditional expectation, we get

$$\begin{aligned}
&\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{-rT}\left(\left(S_0 e^{-T\phi(1)} e^{X_T}\right) e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_t} - K\right)^+ \middle| X_T\right]\right]
\end{aligned}$$

Where the conditional expectation can be interpreted in the Black Scholes model as

a closed formula, leading to the final result.□

Similar results for Binary option

Example 3 *Merton model (1973) with jumps U driven by a Poisson process lognormally distributed* $(\ln(1+U) \sim Normal)$

$$P^i = e^{-\theta(1+c)T} \sum_{i=1}^{+\infty} \frac{(\theta(1+k)T)^n}{n!} P^i_{BS}(S_0, K, T, r_n, \sigma_n) \tag{4.14}$$

with

$$\begin{aligned} c &= \mathbb{E}[J] = e^{m+v^2/2} - 1 \\ r_n &= r - \theta c + \frac{n\left(m + v^2/2\right)}{T} \\ \sigma_n &= \left(\sigma^2 + \frac{nv^2}{T}\right)^{1/2} \end{aligned}$$

with $i = c$ or p corresponding to a call or put option.

Proof: $U + 1$ follows a lognormal law with mean m and volatility v^2, $\mathbb{E}[U] =$

$e^{m+v^2/2} - 1.$

$$\phi(1) = c\theta$$

using the pricing formula (13)

$$P^i = \sum_{i=0}^{+\infty} \mathbb{E}\left[P_{BS}^i\left(S_0 e^{-T\phi(1)} \exp\left(\ln\left(\prod_{j=1}^{n}(1+U_j)\right)\right), K, T, r, \sigma\right)\right] e^{-\theta T}\frac{(\theta T)^n}{n!} \tag{4.15}$$

The random variables $(1+U_i)_{i\in\mathbb{N}^*}$ are independently distributed, with a lognormal distribution with mean m and volatility v^2. Their product $\prod_{j=1}^{n}(1+U_j)$ follows a lognormal distribution with mean nm and volatility nv^2.

Denoting by g a centered normalized normal distribution $(g \sim N(0,1))$, by $S = S_0 e^{-Tc\theta}$, the equation (15) can be rewritten as

$$P^i = \sum_{i=0}^{+\infty} \mathbb{E}\left[\begin{array}{c} \varepsilon S e^{nm+\sqrt{n}vg} N\left(\varepsilon\left(\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\sqrt{n}vg}{\sigma\sqrt{T}} + \frac{\sigma}{2}\sqrt{T}\right)\right) \\ -\varepsilon K e^{-rT} N\left(\varepsilon\left(\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\sqrt{n}vg}{\sigma\sqrt{T}} - \frac{\sigma}{2}\sqrt{T}\right)\right) \end{array}\right] e^{-\theta T}\frac{(\theta T)^n}{n!} \tag{4.16}$$

It can be shown that for $g \sim N(0,1)$ and for every $a, b, c \in \mathbb{R}$,

$$\mathbb{E}\left[e^{ag} N(bg+c)\right] = e^{a^2/2} N\left(\frac{c+ab}{\sqrt{1+b^2}}\right) \tag{4.17}$$

Using the equation (17), we get

$$\mathbb{E}\left[\begin{array}{c}\left(\varepsilon S e^{nm+\sqrt{n}vg}\right) \\ N\left(\varepsilon\left(\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\sqrt{n}vg}{\sigma\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)\right)\end{array}\right]$$

$$= \varepsilon S e^{nm+\frac{nv^2}{2}} N\left(\varepsilon\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\frac{\sigma^2}{2}T+nv^2}{\sigma\sqrt{T}\sqrt{1+\frac{nv^2}{\sigma^2 T}}}\right)$$

and

$$\mathbb{E}\left[\varepsilon K e^{-rT} N\left(\varepsilon\left(\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\sqrt{n}vg}{\sigma\sqrt{T}}-\frac{\sigma}{2}\sqrt{T}\right)\right)\right]$$

$$= \varepsilon K e^{-rT} N\left(\varepsilon \frac{\ln\left(\frac{S}{e^{-rT}K}\right) + nm - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}\sqrt{1+\frac{nv^2}{\sigma^2 T}}}\right)$$

which leads to the result.□

4.2.3 Pricing with Fourier Transform

We denote by $F(f)$ the Fourier transform of a function.

A lemma on Fourier transform function Let f be a function $f : \mathbb{R} \to \mathbb{R}$, continuous, bounded, element of $\mathcal{L}^1(\mathbb{R})$ so that $F(f)$ belongs to $\mathcal{L}^1(\mathbb{R})$. We have

$$\mathbb{E}\left[f(X_t)\right] = \left(F^{-1}\left(F\widetilde{f}(.)\, e^{T\psi(.)}\right)\right)(0) \tag{4.18}$$

with ψ the Fourier exponent as defined above and $F\widetilde{f}(.)$ the Fourier transform of the function $\widetilde{f} : \widetilde{f}(x) = f(-x)$

Proof: Let g denote the convolution product of $\widetilde{f}$ with dp_{X_T}

$$g(y) = \int_{-\infty}^{+\infty} \widetilde{f}(y-x)\, dp_{X_T}(x)$$

g is well defined since f and $\widetilde{f}$ is bounded. g is integrable since

$$\begin{aligned}
\int_{-\infty}^{+\infty} |g(y)|\, dy &\leq \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} \left|\widetilde{f}(y-x)\right| dp_{X_T}(x)\, dy \\
&\leq \int_{x=-\infty}^{+\infty} \left(\int_{y=-\infty}^{+\infty} \left|\widetilde{f}(y-x)\right| dy \right) dp_{X_T}(x) \\
&\leq \left\|\widetilde{f}\right\|_{\mathcal{L}^1(\mathbb{R})} \int_{x=-\infty}^{+\infty} dp_{X_T}(x) \\
&\leq \|f\|_{\mathcal{L}^1(\mathbb{R})}
\end{aligned}$$

The expectation of $f(X_t)$ can be seen as the value in zero of this convolution product since

$$\begin{aligned}
\mathbb{E}[f(X_t)] &= \int_{-\infty}^{+\infty} f(x)\, dp_{X_T}(x) \\
&= \int_{-\infty}^{+\infty} \widetilde{f}(-x)\, dp_{X_T}(x) \\
&= g(0)
\end{aligned}$$

□

This means that we can use the Levy-Khintchine exponent. From a numerical point of view, many efficient methods to compute convolution products with FFT algorithm

However, it is not straightforward since the payoff function does not belong to $\mathcal{L}^1(\mathbb{R})$!
Solution: use a truncated version of the Payoff function like for a call option:

$$f^c_{M_c}(x) = (e^x - K)^+ \left(1_{\{x \leq M_c\}} + e^{-\alpha(x-M_c)} 1_{\{x > M_c\}}\right) \tag{4.19}$$

which converges to the standard payoff:

$$\lim_{|M| \to +\infty} \mathbb{E}\left[e^{-rT} f^i_{M_i}\right] = \mathbb{E}\left[e^{-rT} f^i\right] \tag{4.20}$$

This leads to a methodology for pricing options.

5 Volatility smile

5.1 Preliminary

Carr (1997) If f belongs to $C^2\left(\mathbb{R}^{+*}\right)$ (set of functions twice differentiable with continuous second order derivative function), then for every $x \in \mathbb{R}^+$, for every $\kappa \in \mathbb{R}^+$

$$f(x) = \int_0^{\kappa} (x-u)^+ f''(u)\, du + \int_{\kappa}^{+\infty} (u-x)^+ f''(u)\, du$$

5.2 Estimation of the Levy process

Denoting by $Put(u)$ (respectively $Call(u)$) the price of a put (respectively a call) on S_T with strike u, we can calculate the Levy-Laplace exponent as

$$\begin{aligned}
&\phi(\lambda) \\
&= \frac{1}{T}\ln\left(\frac{\lambda(\lambda-1)}{S_0^{\lambda}}\int_0^{+\infty} e^{rT} Put(u)\, u^{(\lambda-2)} du\right) - \lambda\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2\lambda^2}{2} \\
&= \frac{1}{T}\ln\left(\frac{\lambda(\lambda-1)}{S_0^{\lambda}}\int_0^{+\infty} e^{rT} Call(u)\, u^{(\lambda-2)} du\right) - \lambda\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2\lambda^2}{2}
\end{aligned}$$

Proof: The function $f : x \to x^{\lambda}$ belongs to $C^2(\mathbb{R}^{+*})$. The preliminary 5.1 leads to

$$\mathbb{E}\left[\frac{S_T^{\lambda}}{\lambda(\lambda-1)}\right] = \int_0^{\kappa} \mathbb{E}\left[(S_T-u)^+ u^{(\lambda-2)}\right] du + \int_{\kappa}^{+\infty} \mathbb{E}\left[(u-S_T)^+ u^{(\lambda-2)}\right] du$$

using the two limiting case $\kappa = 0$ and $\kappa = +\infty$, we get

$$\begin{aligned}\mathbb{E}\left[\frac{S_T^{\lambda}}{\lambda(\lambda-1)}\right] &= \int_0^{+\infty} e^{rT} Put(u)\, u^{(\lambda-2)} du \\ &= \int_0^{+\infty} e^{rT} Call(u)\, u^{(\lambda-2)} du\end{aligned}$$

□

This is a way of inferring from option prices the Levy-exponent. Then, from that, we can use the different methods to reprice options in a consistent way.

6 Conclusion

- Levy processes are becoming an important building block of the option pricing literature.
- Their generality and complexity in terms of measure and integro-differential equation is not an obstacle to their use.
- There is still open questions on the appropriate measure, their empirical fit to data and their application for exotic option where numerical methods are required.