Optimal Malliavin Weighting Function for the Computation of the Greeks

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Abstract

This paper reexamines the Malliavin weighting functions introduced by Fournié et al. (1999) as a new method for efficient and fast computations of the Greeks. Reexpressing the weighting function generator in terms of its Skorohod integrand, we show that these weighting functions have to satisfy necessary and sufficient conditions expressed as conditional expectations. We then derive the weighting function with the smallest total variance. This is of particular interest as it bridges the method of Malliavin weights and the one of likelihood ratio, as introduced by Broadie and Glasserman (1996). The likelihood ratio is precisely the weighting function with the smallest total variance. We finally examine when to use Malliavin method and when to prefer finite difference.

1 Introduction

The growing emphasis on risk management issues as well as the development of more and more complicated financial products have urged to develop efficient techniques for the computation of price sensitivities with respect to model parameters. Furthermore, these Greeks are not only very useful for the risk management and hedging strategy but also for the pricing of the product. Price sensitivities contribute directly to the price quote since the bid and ask spread is often taken as a proportion of some Greeks. They are also used to estimate the pricing error as they show the impact of a parameter that may be inappropriate or vary during the life of the product. Last but not least, the computation is not only done as the trader level or book level but also at the firm level, especially for the global computation of VAR and credit adjustment, leading to raising concern about computational time. Unfortunately, in many cases, these risk ratios can not be expressed as closed form and require numerical methods. One of the most flexible method is the Monte Carlo one.

However, a straightforward simulation, spiced up by various variance reduction techniques, is inefficient in the case of options with discontinuous payoff. This comes from the way the Greeks are computed. Defined mathematically as derivatives of the price function with respect to specific parameters, the Greeks are traditionally estimated by means of a finite difference approximation. One calculates a price, bumps the parameters and reprices it. This approximation contains two errors: one on the approximation of the

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derivative function by means of its finite difference and another one on the numerical computation of the expectation.

To overcome this poor convergence rate, many methods have been suggested, out of which two have emerged as being very efficient. On the one hand, Broadie and Glasserman (1996) advocated to differentiate the density function, introducing the likelihood ratio. On the other hand, Fournié, Lasry, Lions, Lebuchoux, Touzi (1999) suggested to smoothen the function to be estimated by an integration by parts. The approach, similar to a result of Elworthy (1992) consists in shifting the differential operator from the payoff functional to the diffusion kernel, introducing a weighting function. Their work proved that the common Greeks can be rewritten as an expected value of the payoff times a weighting function.

$$Greek = \mathbb{E}^Q \left[e^{-\int_0^T r_s ds} f(X_T).weight \right] \tag{1.1}$$

Their results were given for particular examples of weighting functions. However, many points remained unsolved. What is the link between the likelihood ratio of Broadie and Glasserman and the Malliavin weighting function of Fournié et al.? What are the different admissible weighting functions? Can we determine some necessary and sufficient conditions for a weighting function to be satisfied? What is the best weighting function, according to the total variance of the expectation to be calculated?

The contributions of this paper is precisely to provide an answer to these questions. To do so, we introduce the weighting function generator defined as its Skorohod integrand. We show that these weighting functions can be characterized by necessary and sufficient conditions on their generator, by means of some conditional expectation. We then provide a closed formula for the weighting function of minimal variance. This enables us to find a relationship between the approach of Broadie and Glasserman and the one of Fournié et al. We prove that the weighting function with minimal variance is precisely the likelihood ratio of Broadie and Glasserman for diffusions with explicit density function. We finally describe when to and when not to apply the Malliavin weighting method.

The remainder of this article is organized as follows. In section 2, we review briefly the different methods. Section 3 shows how to derive the necessary and sufficient conditions for the weighting function generator. Section 4 examines the case of the weighting function with minimal variance and shows that this particular solution is precisely the likelihood ratio. We finally examine, in section 5, some numerical examples of Malliavin calculus.

2 Previous Methods

2.1 Convergence results

Theoretical results about the convergence of price sensitivities via Monte Carlo are well known. Glynn (1989) showed that the quality of this approximation was depending on the way of approximating the derivative function: forward, central or even backward difference scheme. In the case of the forward $((P(x+\varepsilon) - P(x))/\varepsilon)$ or backward $((P(x) - P(x-\varepsilon))/\varepsilon)$ difference scheme, if the simulation of the two expectations is drawn independently, he proved that the best theoretical convergence rate is $n^{-1/4}$. As of the central $((P(x+\varepsilon) - P(x-\varepsilon))/2\varepsilon)$ difference scheme, the optimal rate is $n^{-1/3}$. When taking common random numbers, this optimal rate becomes $n^{-1/2}$. This is the best to be expected by standard Monte Carlo simulation. However, the finite difference method is inefficient when dealing with discontinuous payoffs. This restriction applies to many of the exotic options such as digital, corridor, Asian and lookback options.

2.2 Brief review of previous works

To overcome the poor convergence rate for exotic options, Curran (1994) suggested to take the differential of the payoff function inside the expectation. Later, Broadie and Glasserman (1996) suggested to take the derivative of the density function and introduced the likelihood ratio defined as $\frac{\partial}{\partial\theta}\ln p\left(X_T,\theta\right)$ where $p\left(X_T,\theta\right)$ represents the density function of an underlying asset with a parameter θ. This leads to a smoothened expression for the Greek where the payoff function is not differentiated:

$$\frac{\partial}{\partial\theta}\mathbb{E}\left[f\left(X_T\right)\right]=\mathbb{E}\left[f\left(X_T\right)\frac{\partial}{\partial\theta}\ln p\left(X_T,\theta\right)\right] \tag{2.1}$$

This method has however the disadvantage to require an explicit expression of the density function. To avoid the need of a closed formula of the density function, Fournié et al.(1999) introduced a new method based on Malliavin calculus. This method requires only the definition of the diffusion equation. Its problem is the flexibility of the weighting functions as well as the exact derivation of the weighting functions. Avellaneda et al . (2000) suggested another method for deriving the weighting function, inspired by Kullback Leibler relative entropy maximization. The weighting function is obtained by perturbing the vector of probability obtained by the Monte Carlo simulation. Kohatsu-Higa (2000) studied the property of the of variance reduction implied by the Malliavin weighting functions. Pikovsky (2000) gave sufficient conditions on the diffusion parameters to be able to interchange the expectation and derivation operator. There has been many extensions to the initial work: see Benhamou (2000a) for more results on the Asian option, Gobet and Kohatsu-Higa (2001) for Malliavin weights for barrier and lookback options, Fournié, Lasry, Lebuchoux and Lions (2000) for results on conditional expectations and Lions and Régnier (2000) for the case of the American options.

3 Characterization of the weighting functions

3.1 Mathematical framework

We consider a continuous time trading economy with a finite horizon $t\in[0,T]$, with a complete market in which there exists a risk neutral probability measure Q uniquely defined by the no-arbitrage condition. The uncertainty in this economy is classically modeled by a complete probability space (Ω,F,Q). The information evolves according to the augmented filtration $\{F_t,t\in[0,T]\}$ generated by a standard one dimensional standard Wiener process $(W_t)_{t\in[0,T]}$. To avoid heavy notations, and for clarity reason, we present our results in one dimension. However, our results can easily be extended to the multi-dimensional case. The evolution of the underlying price, Ito process $(X_t)_{t\in[0,T]}$, is described by a general stochastic differential equation (SDE):

$$dX_t=b\left(t,X_t\right)dt+\sigma(t,X_t)dW_t \tag{3.1}$$

with the initial condition $X_0=x$, $x\in\mathbb{R}$. The function $b:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ represents the determinist drift of our process and the function $\sigma:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ its volatility. The risk free interest rate is denoted by $r\left(t,X_t\right)$. We assume that the functions b and σ are C_1^b: continuously differentiable with bounded derivatives. This implies in particular that they satisfy Lipschitz conditions[1], i.e., there exists a constant $K<+\infty$ such that

$$\left|b\left(t,x\right)-b\left(t,y\right)\right|+\left|\sigma\left(t,x\right)-\sigma\left(t,y\right)\right| \leq K\left|x-y\right| \tag{3.2}$$

$$\left|b\left(t,x\right)\right|+\left|\sigma\left(t,x\right)\right| \leq K\left(1+\left|x\right|\right) \tag{3.3}$$

[1] Inequalities (3.2) and (3.3) ensure the existence and unicity of a continuous, strong solution of the SDE (3.1) with its initial condition.

We denote by X_t^x the continuous, strong solution X_t starting at x. We assume as well that the diffusion function $\sigma(t,x)$ is uniformly elliptic:

$$\exists \epsilon > 0, \quad \forall t \in [0,T], \forall x \in \mathbb{R} \quad |\sigma(t,x)| \geq \epsilon \tag{3.4}$$

We denote by $(Y_t)_{t\in[0,T]}$ the first variation process of $(X_t)_{t\in[0,T]}$, which is characterized as the unique strong continuous solution of the linear stochastic differential equation (3.5) with initial condition $(Y_{t=0} = 1)$:

$$\frac{dY_t}{Y_t} = b'(t, X_t)dt + \sigma'(t, X_t)dW_t \tag{3.5}$$

where the prime stands for the derivatives with respect to the second variable. We can show that the first variation process is the derivative of $(X_t)_{t\in[0,T]}$ with respect to x, $\left(Y_t = \frac{\partial}{\partial x} X_t\right)$. Malliavin calculus theory (see Nualart (1995) Theorem 2.3.1 page 110) proves that the Malliavin derivative can be written as an expression of the first variation process and the volatility function:

$$D_s X_t = Y_t Y_s^{-1} \sigma(s, X_s) 1_{\{s \leq t\}} a.s. \tag{3.6}$$

To be as general as possible, we assume that our payoff is depending on a finite set of payment dates: $t_1, t_2, ..., t_m$ with the convention that $t_0 = 0$ and $t_m = T$. The price $P(x)$ of the contingent claim given an initial value of the underlying price x is traditionally computed as the expectation under the risk neutral probability measure of discounted future cash flow:

$$P(x) = \mathbb{E}_x^Q \left[e^{-\int_0^T r(s,X_s)ds} f\left(X_{t_1}, X_{t_2}, ..., X_{t_m}\right) \right]$$

with the traditional shortcut notation $\mathbb{E}_x^Q[.] = \mathbb{E}^Q[.|X_0 = x]$. The function $f : \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R} \to \mathbb{R}$ denotes the payoff, and is supposed to be first order differentiable with a derivative with polynomial growth. We denote by F the discounted payoff $F = e^{-\int_0^T r(s,X_s)ds} f\left(X_{t_1}, X_{t_2}, ..., X_{t_m}\right)$. If we need to specify that the underlying is a function of the initial value x, we denote the discounted payoff by F^x.

3.2 Generators of a Weighting function

The work of Fournié et al. proved that any Greeks could be written as an expected value of the payoff times a weighting function (1.1). This comes from an integration by parts formula given by the Malliavin calculus theory. One can also re-expressed that in the dual form.

Let $C_p^\infty(\mathbb{R}^m)$ be the set of C^∞ functions $f : \mathbb{R}^m \to \mathbb{R}$ with polynomial growth and derivatives of all orders with polynomial growth. Let S be the set of real random variable of the form $f\left(X_{t_1}, X_{t_2}, ..., X_{t_m}\right)$ and $\mathbb{D}^{1,2}$ the Banach space which is the completion of S with respect to the norm:

$$\|F\|_{1,2} = \left(\mathbb{E}\left[F^2\right]\right)^{1/2} + \left(\sum_{i=1}^m \mathbb{E}\left[\left(\int_0^T \frac{\partial}{\partial x_i} f\left(X_{t_1}, X_{t_2}, ..., X_{t_m}\right) D_s X_{t_i}\right)^2\right]\right)^{1/2}$$

The adjoint of the closed unbounded operator $D : \mathbb{D}^{1,2} \to L^2([0,T] \times \Omega)$ is usually denoted by δ and is called the Skorohod integral. Its domain is characterized as the set of measurable process $u \in L^2([0,T] \times \Omega)$ such that there exists a positive constant C that may depend on u such that

$$\mathbb{E}\left(\int_0^T D_t F u_t dt\right) \leq C \|F\|_{1,2}$$

Writing the weighting function *weight* as a Skorohod integral, we call weighting function generator or generator w the Skorohod integrand

$$weight = \delta(w) \tag{3.7}$$

To ensure the existence of the Skorohod integral, we impose that the weight function is L^2 integrable:

$$\mathbb{E}\left[weight^2\right]^{1/2} < \infty \tag{3.8}$$

The point of view taken here is to characterize the weighting function by its generator. We have the following theorem, proved in this paper for the delta. Extension to other Greeks is straightforward and can be found in Benhamou (2000b)

Theorem 1 *Necessary and sufficient conditions for the weighting functions generator*
There exist necessary and sufficient conditions for a function w to serve as a generator. The first condition is the Skorohod integrability of this function. The second condition, summarized in table 1, depends only on the underlying diffusion characteristics and is independent from the payoff function.

Proof: in the appendix section.□

Greeks	**Necessary and Sufficient conditions on the Malliavin Weights**
delta	(C1.1) : $\mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}\left[DC(0,T)\,Y_{t_i}\int_0^{t_i}\frac{\sigma(t,X_t)}{Y_t}w^{delta}(t)\,dt\right] = \mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}\left[DC(0,T)\,Y_{t_i}\right]$ (C1.2) : $\mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}\left[DC(0,T)\int_{0=t\le s\le T} r'(s,X_s)\frac{Y_s\sigma(t,X_t)}{Y_t}w^{delta}(t)\,dtds\right] = \mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}\left[DC(0,T)\int_0^T r'(s,X_s)\,Y_s ds\right]$

Table 1: Necessary and Sufficient conditions for the Weighting Function Generators of a delta

In the table 1, we denote by $\mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}$ the conditional expectation with respect to $X_{t_1},...,X_{t_m}$, i.e. $\mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}[.] = \mathbb{E}^Q_x[|X_{t_1},...,X_{t_m}]$. We denote by $DC(0,T)$ the stochastic discount factor term $e^{-\int_0^T r(u,X_u^x)du}$.

4 The minimal variance weighting function

4.0.1 Optimal weighting function

To find the weighting function with the minimal variance, we just need to re-interprate the formula for the Greeks. It is in fact the scalar product of the weighting function with a function $X_{t_1},...,X_{t_m}$ measurable. Without more information, we know using a standard projection theorem, that the weighting function with minimal variance is the conditional expectation of any weighting function with respect to the variables $X_{t_1},...,X_{t_m}$. This results is stated in the following proposition.

Proposition 2 *The weighting function with minimal variance denoted by π_0 is the conditional expectation of any weighting function with respect to the variables $X_{t_1},...,X_{t_m}$*

$$\pi_0 = \mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}[weight] \tag{4.1}$$

Proof: Let π be any weighting function and π_0 its conditional expectation with respect to $X_{t_1},...,X_{t_m}$ ($\pi_0 = \mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}[weight]$) which is uniquely defined. The Greek can be expressed as the expected value of the scalar product of the discounted payoff denoted by F with the weighting function π

$$Greek = \mathbb{E}[F.\pi]$$

One can see that the variance V of the estimate $F.\pi$, $V = \mathbb{E}\left[(F.\pi - Greek)^2\right]$, can be expressed with respect to π_0

$$\begin{aligned} V &= \mathbb{E}\left[(F.(\pi - \pi_0))^2\right] + \mathbb{E}\left[(F.\pi_0 - Greek)^2\right] \\ &\quad + 2\mathbb{E}\left[(F.(\pi - \pi_0)).(F.\pi_0 - Greek)\right] \end{aligned}$$

The last term in the equation above is equal to zero since

$$\begin{aligned} \mathbb{E}\left[(F.(\pi - \pi_0)).(F.\pi_0 - Greek)\right] &= \mathbb{E}\left[\mathbb{E}\left[(F.(\pi - \pi_0)).(F.\pi_0 - Greek)\,|X_{t_1},...,X_{t_m}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[(F.(\pi - \pi_0))\,|X_{t_1},...,X_{t_m}\right].(F.\pi_0 - Greek)\right] \\ &= 0 \end{aligned}$$

where we have used first the fact that $(F.\pi_0 - Greek)$ and F are $X_{t_1},...,X_{t_m}$ measurable.□

This is a strong result. It indicates that the best weighting function should always be the one $X_{t_1},...,X_{t_m}$ measurable. It indicates as well that without any more specification on the pay-off function, the variance is lower-bounded by the variance of this particular weighting function π_0. To improve this lower bound, we need to have additional information on the payoff function. It is worth noticing that the set of the Malliavin weighting function is defined by conditions which are independent from the payoff function. However, the optimal solution in the sense of the total variance crucially depends from the payoff's state variables.

4.1 Link with the likelihood ratio

In the case of an explicit density function, Broadie and Glasserman (1996) showed that the Greek can be written as the expectation of the payoff function times the likelihood ratio. Since the Ito integral and Skorohod integral coincide over the set of adapted functions, the likelihood ratio can be seen as a Malliavin weighting function. Moreover, The likelihood ratio is a function of the different state variables of the payoff function or more precisely is $X_{t_1},...,X_{t_m}$ measurable. This indicates that the likelihood ratio is precisely the weighting function with minimal variance. Unfortunately, the derivation of the likelihood ratio requires the explicit knowledge of the density function while the Malliavin theory enables us to derive the optimal weight with much less information.

5 Numerical results

5.1 The failure of finite difference for discontinuous payoff options

The variance of a finite difference scheme depends on the variance of $(P(x+\varepsilon) - P(x))/\varepsilon$, proportional to

$$Var(P(x)) + Var(P(x+\varepsilon)) - 2Cov(P(x+\varepsilon), P(x))$$

Common random numbers will improve the convergence as it will increase the covariance between the bumped price $P(x+\varepsilon)$ and the standard price $P(x)$. Furthermore, the above equation shows that the method relies on the fast mean-square convergence of $P(x+\varepsilon)$ to $P(x)$. The theoretical optimal rate of convergence of $n^{-1/2}$ unfortunately does not apply in all cases. It is easy to see that the more discontinuous the payoff function is, the slower the mean-square convergence of $P(x+\varepsilon)$ to $P(x)$ is. For example, let us examine the case of the digital call, an option paying 1 in the case of an underlying above the strike $X_T > K$ and zero elsewhere. The difference between the shifted digital call $P(x+\varepsilon)$ and the regular digital call $P(x)$ is given by a probability times a discount rate squared:

$$\mathbb{E}\left[|P(x+\varepsilon) - P(x)|^2\right] = e^{-2rT} P\left[X_T < K < X_T(\varepsilon)\right]$$

Assuming an homogeneous underlying process, $X_T(\varepsilon) = X_T * \left(1 + \frac{\varepsilon}{x}\right)$, it leads to a convergence rate of ε for this probability. Writing with Landau notation, we get that the convergence of $P(x+\varepsilon)$ to $P(x)$ is only linear in ε :

$$\mathbb{E}\left[|P(X_0+\varepsilon) - P(X_0)|^2\right] = O(\varepsilon)$$

On the contrary, in the case of the plain vanilla call option, it can be shown (see for example Broadie and Glasserman (1996)) that for the geometric Brownian motion, the convergence rate is of ε^2

$$\begin{aligned}\mathbb{E}\left[|P(x+\varepsilon) - P(x)|^2\right] &\leq \mathbb{E}\left[|X_T(\varepsilon) - X_t|^2\right] \\ &\leq \varepsilon^2 \mathbb{E}\left[e^{(r-\mu)T + \sigma\sqrt{T}Z}\right]\end{aligned}$$

where Z is a normal variable $N(0,1)$, leading to

$$\mathbb{E}\left[|P(x+\varepsilon) - P(x)|^2\right] = O\left(\varepsilon^2\right)$$

This is why the methodology of finite difference under-performs for all discontinuous type options like simple digital, corridor (option which pays 1 if the underlying at time T is inside an interval $L < X_T < H$), barrier option and so on.

5.2 Characteristic of Malliavin weighting method

We remind important characteristics of Malliavin weights:

- All Greeks can be written as the expected value of the payoff times a weight function. The weight functions are independent from the payoff function. This has two implications.
  - First, the Malliavin method will comparatively (to finite difference) increased its efficiency for discontinuous payoff options. As a rule of thumb, the Malliavin method is appropriate for option for which the mean-square convergence of a shifted option $P(X_0+\varepsilon)$ to the normal one $P(X_0)$ is linear in ε. This is the case of any option with a payoff expressed as a probability that a certain event occurs conditionally to the underlying level at a certain time (case of any binary and corridor option).
  - Second, no extra computation is required for other payoff function as long as the payoff is a function of the same points of the Brownian trajectory since the weight does not change.
- There is an infinity of solutions for the generator function. However, the optimal weighting function is the one $X_{t_1}, ..., X_{t_m}$ measurable. This means in practice that the weight functions will be expressed with the same points of the Brownian motion trajectory as the option payoff, therefore requiring no extra points computation.
- The weighthing function smoothens the function to simulate (as the payoff function does not require to be numerically differentiated) but introduces some extra noise. It smoothens twice the payoff function in the case of the gamma as it reduces a second order differentiation to no differentiation, leading to high efficiency for the simulation of the gamma (see figure1 for the comparative efficiency of the Malliavin method in the case of the gamma of a corridor option). It introduces a lot of noise in the simulation as the weighting function explodes for short maturities options.
- For homogeneous model, like Black Scholes, various tricks can be used to derive one Greeks from the others. In particular, the computation of the vega is similar to the gamma since there is a

direct proportionality between the gamma and vega. This has two implications: first, the simulation of gamma and vega can be done at once and second, the convergence of the vega is very similar to the one of the gamma. The similarity of the vega with the gamma can also be understood by the fact that the the vega is a compound differentiation $\frac{\partial}{\partial\sigma} f\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}\right) = f'\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}\right) \frac{\partial}{\partial\sigma} e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}$, which can be re-interpreted by an integration by parts as a second order derivatives.

- The Malliavin method leads to weighting functions which are roughly (polynomial) functions of the Brownian motion. The variance of the weighting function increases for high values of the Brownian motion. This implies that if the payoff function is very small for high value of the Brownian motion, the variance is going to be low. This indicates that Malliavin formulae are more efficient for put than call options. Two remarks should be made. First, it is more appropriate to use the put-call parity and therefore to calculate Greeks only for a put, second, one should use a localization of the Malliavin weight only at the discontinuity of the payoff and elsewhere avoid introducing extra noise with the Malliavin weight.

In order to illustrate these remarks, we show two simulations done for the gamma of a European corridor and call option in a Black Scholes model. Formulae for Malliavin weight can be found in Fournié et al. (1999) or Benhamou (2000b) and are summarized for the European option in table 2. Figure 1 is a school case of an appropriate use of Malliavin method. The payoff of a corridor option has two discontinuities, the mean square convergence of the bumped price is only linear in ε and the Malliavin method smoothens twice the Greek to simulate in the case of the gamma. Figure 2 is an example of inappropriate use of Malliavin method. The mean square convergence of the bumped price is quadratic, the payoff is not discontinuous, it is only its derivative function that has only one discontinuity at the strike. The Malliavin method introduces extra noise in the simulation with the weight to simulate. The call put parity was not used, therefore creating high variance for high values of the Brownian motion.

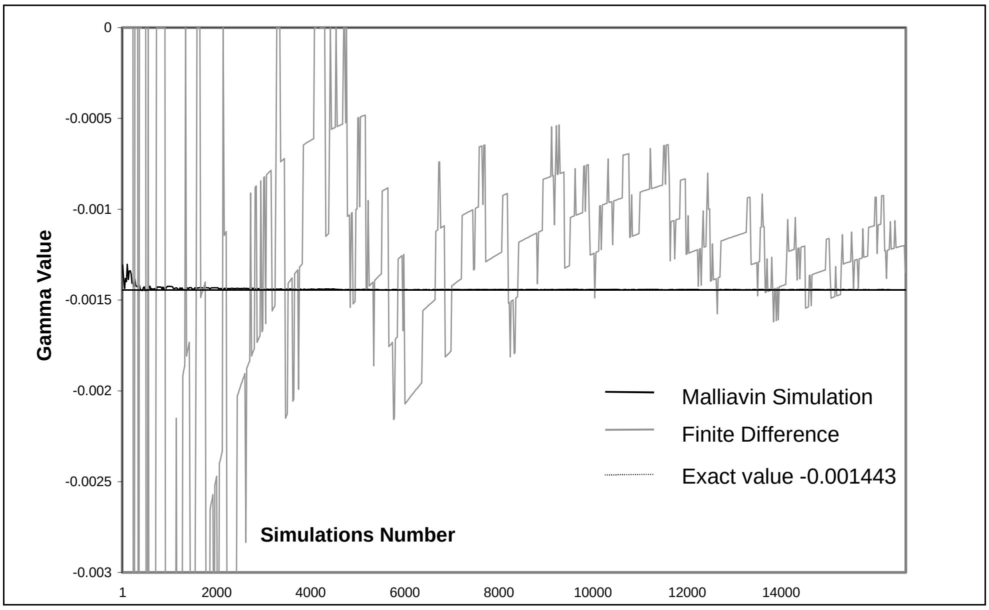

Figure 1: Efficiency of the Malliavin weighted scheme for the computation of the gamma of a Corridor option. The parameters of this option are: $S_0 = 100, \quad r = 5\%, \quad \sigma = 15\%, \quad T = 1year, \quad S_{\min} = 95, \quad S_{\max} = 105$

Greek Name	Malliavin weight
delta	$\frac{W_T}{T\sigma x}$
gamma	$\frac{1}{\sigma T^2 x}\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)$
vega	$\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T$
rho	$\frac{W_T}{\sigma} - T$

Table 2: Optimal Malliavin weighting function for a European option in a Black Schoes model

6 Conclusion

In this paper, we have shown that the weighting functions as introduced by Fournié, Lasry, Lions, Lebuchoux, Touzi (1999) can be characterized by necessary and sufficient conditions given as conditional expectation. We have derived the weighting function with minimal variance expressed a conditional expectation of any weighting function with respect to the different state variable of the payoff function. We have given the relationship between the likelihood ratio of Broadie and Glasserman (1996) and the Malliavin weighting function of Fournié et al. (1999). We have given some general indications for an appropriate use of the Malliavin weights methods.

Extension to this work encompasses the impact of a localization of the Malliavin weighted method as suggested in Fournié et al. (1999) and later in Benhamou (2000a).

A Appendix

We provide the proof for the delta. Results and proofs for the gamma, vega and rho follow similar methods and can be found in Benhamou (2000b).

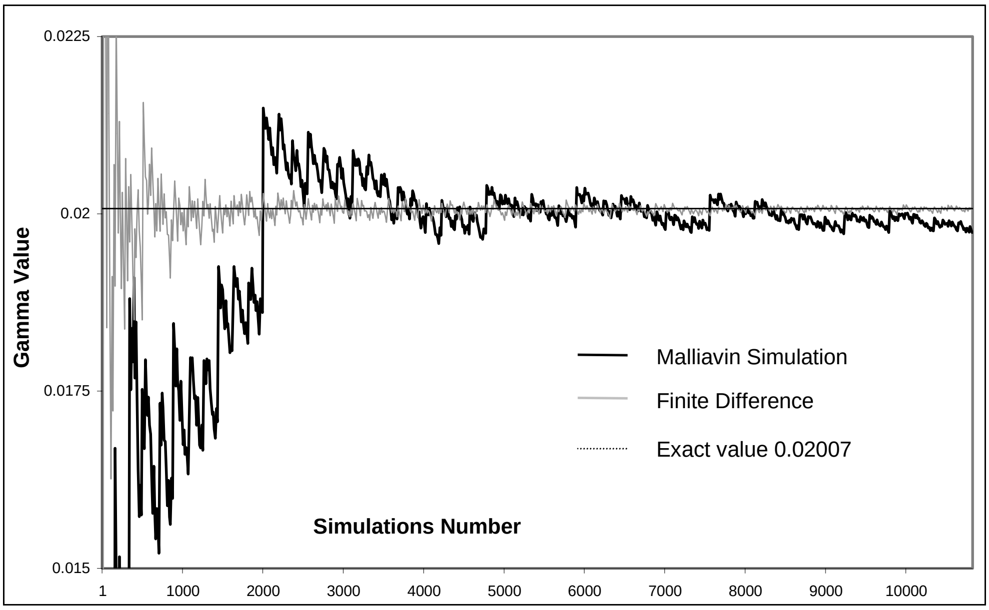

Figure 2: Efficiency of the Malliavin weighted scheme for the computation of the delta of a call option. The parameters are similar to the corridor option with a strike of 100

A.1 Proof of the delta formula

In this section, we prove that the weighting function for the delta should satisfy necessary and sufficient conditions. For the sake of simplicity, we denote in this section w^{delta} by w, and denote by a prime the derivative with respect to the second variable. The part of the proof based on integration by parts is quite short and follows the one of Elworthy (1992). The technical difficulty here is to justify rigorously the use of weaker assumptions. It can be divided into three major steps:

1. first preliminary: weaker conditions on the payoff function f: show that if the result holds for any function of C_K^∞ (set of infinitely differentiable functions with compact support), it also holds for any element of L^2.

2. second preliminary: interchange of the order of differentiation and expectation: show that one can interchange the order of differentiation and expectation.

3. integration by parts:

   (a) necessary condition.

   (b) sufficient condition.

A.1.1 First preliminary: Weaker assumptions

We denote in the following f the payoff function and F the payoff function times the discount factor. The idea of the first technical point is the following: taking f as an element of L^2 is the same as assuming f infinitely differentiable with a compact support. It is based on a density argument using Cauchy Schwartz inequality and the continuity of the expectation operator.

More precisely, let us assume the result is true for any function of C_K^∞ (set of infinitely differentiable functions with compact support). Let f be now only in L^2. Using the density of $C_K^\infty[0,T]$ in L^2, there exits a sequence $(f_n)_{n\in N}$ of C_K^∞ elements that converges to f in L^2. Let's denote $u(x) = \mathbb{E}_x^Q[F]$ and $u_n = \mathbb{E}_x^Q[F_n]$ the prices associated with the discounted payoff functions F and F_n (F and F_n are the function f and f_n times the discount factor function). x is the starting point of the underlying security price. Since L^2 convergence implies L^1 convergence, we know that the set of functions u_n converges simply to the function u.

$$\forall x \in \mathbb{R} \quad u_n(x) \underset{n\to\infty}{\to} u(x)$$

Since the result is true for payoff functions element of C_K^∞, the derivative of the u_n function can be written as the expectation of the discounted payoff function f_n times a suitable "Malliavin" weight $\delta(w)$ defined as the Skorohod integral of a function w:

$$\frac{\partial}{\partial x} u_n(x) = \mathbb{E}_x^Q \left[F_n \delta(w)\right]$$

Let's denote by g the function obtained as the expectation of the discounted payoff function f times the Malliavin weight $\delta(w) : g(x) = \mathbb{E}_x^Q \left[F\delta(w)\right]$. By Cauchy Schwartz inequality

$$\left| g(x) - \frac{\partial}{\partial x} u_n(x) \right| = \left| \mathbb{E}_x^Q \left[(F - F_n)\, \delta(w)\right] \right| \leq h(x)\, \epsilon_n(x) \tag{A.1}$$

with

$$h(x) = E_x^Q \left[(\delta(w))^2 \right]^{1/2} \quad \epsilon_n(x) = E_x^Q \left[(F - F_n)^2 \right]^{1/2}$$

By definition, the L^2 convergence of u_n means $\epsilon_n(x)$ converges simply to zero as n tends to infinity. Therefore we already know that the function sequence $\left(\frac{\partial}{\partial x} u_n\right)_{n\in N}$ converges simply to the function g. By property of Lebesgue compacity and the fact that the functions F and F_n are continuous and that $h(x)$ is bounded (non-explosive condition (3.8)), inequality (A.1) proves that this convergence is uniform on any compact subsets K of $\mathbb{R}$..

We conclude using the fact that if a sequence of functions $(u_n)_{n\in\mathbb{N}}$ converges simply to a function u and the sequence of function's derivative $\left(\frac{\partial}{\partial x} u_n\right)_{n\in\mathbb{N}}$ converges uniformly to a function g on any compact subsets of $\mathbb{R}$, the limit function u is continuously differentiable with its derivative equal to the limit function of the sequence of function's derivative $\left(\frac{\partial}{\partial x} u_n\right)_{n\in\mathbb{N}}$ leading to the final result:

$$\frac{\partial}{\partial x} E_x^Q [F] = \mathbb{E}_x^Q \left[F\delta(w)\right]$$

□

A.1.2 Second preliminary: Interchanging the order of expectation and differentiation

The second technical point is to show that we can interchange the order of expectation and differentiation (using the dominated convergence theorem).

More precisely, since because of the first preliminary, f is assumed to be element of C_K^∞ and therefore is continuously differentiable with bounded derivative, we have

$$\frac{F^{x+h} - F^x}{\|h\|} - \frac{\left\langle \frac{\partial}{\partial x} F, h \right\rangle}{\|h\|} \underset{\|h\|\to 0.}{\to} 0 \quad a.s.$$

An elementary calculation gives us

$$\frac{\partial}{\partial x} F = \begin{pmatrix} e^{-\int_0^T r(s, X_s^x) ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right) \frac{\partial}{\partial x} X_{t_i}^x \\ -F \int_0^T r'(s, X_s^x) \frac{\partial}{\partial x} X_s^x ds \end{pmatrix}$$

Since f has bounded derivative,first, $\frac{\left\langle \frac{\partial}{\partial x} F, h \right\rangle}{\|h\|}$ is uniformly integrable in h and second, by Taylor Lagrange theorem,

$$\left\| \frac{F^{x+h} - F^x}{\|h\|} \right\| \leq M \sum_{i=1}^m \frac{\left\| X_{t_i}^{x+h} - X_{t_i}^x \right\|}{\|h\|}$$

Using the result that $\sum_{i=1}^{m} \frac{\left\| X_{t_i}^{x+h} - X_{t_i}^{x} \right\|}{\|h\|}$ is uniformly integrable in h (See Theorem 2.4 p 362 Chapter IX Stochastic Differential Equations, Revuz and Yor (1994)) leads to the uniform integrability in h of $\left\| \frac{F^{x+h} - F^{x}}{\|h\|} \right\|$

This in turn tells us that $\frac{F^{x+h} - F^{x}}{\|h\|} - \frac{\left\langle \frac{\partial}{\partial x} F, h \right\rangle}{\|h\|}$ is uniformly integrable in h. Since it converges to zero a.s., the dominated convergence theorem gives us that it converges also to zero in L^1. We conclude that

$$\frac{\partial}{\partial x} u\left(X^x\right) = \mathbb{E}_x^Q \left[\frac{\partial}{\partial x} F \right] \tag{A.2}$$

□

A.1.3 Integration by parts:

Necessary condition: In this subsection, we examine the necessary condition to be satisfied by the weighting function. The delta is defined as the derivative of the price function with respect to the initial condition x

$$delta = \frac{\partial}{\partial x} \mathbb{E}_x^Q \left[e^{-\int_0^T r(s, X_s^x) ds} f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x \right) \right] \tag{A.3}$$

Writing the delta in terms of its weighting function generator and using the property of adjoint of the Skorohod integral leads to:

$$\begin{aligned} delta &= \mathbb{E}_x^Q \left[\left\langle e^{-\int_0^T r(s, X_s^x) ds} f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x \right), \delta(w) \right\rangle \right] \\ &= E_x^Q \left[\left\langle D_t \left(e^{-\int_0^T r(s, X_s^x) ds} f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x \right) \right), w(t) \right\rangle \right] \\ &= \mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s, X_s^x) ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x \right) \int_{t=0}^T D_t X_{t_i}^x w(t)\, dt \\ -F \int_{t=0}^T \int_{s=0}^T \frac{\partial}{\partial X} r\left(s, X_s^x\right) D_t X_s w(t)\, ds dt \end{array} \right] \end{aligned}$$

where in the last line, we have used the property of Malliavin derivatives for compound functions and the fact that we only deal here with one dimension processes. Using the relationship between the Malliavin derivative and the first variation process (3.6), we can replace the expression of $D_t X_u$ $u \geq t$ in the equation above, leading to

$$\mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s, X_s^x) ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x \right) \\ \int_0^T Y_{t_i} Y_t^{-1} \sigma(t, X_t^x) w(t)\, 1_{\{t \leq t_i\}} dt \\ -F \int_{s=0}^T \int_{t=0}^T \frac{\partial}{\partial X} r\left(s, X_s^x\right) Y_s Y_t^{-1} \sigma(t, X_t^x) w(t)\, t 1_{\{t \leq s\}} dt ds \end{array} \right]$$

On the other hand, the delta is defined as the derivative of the price function with respect to the initial condition x. Using (3.6) and the second preliminary's results (A.2), we can change the LHS of (A.3)

$$\begin{aligned} & delta \\ &= \mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s, X_s^x) ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x \right) \frac{\partial}{\partial x} X_{t_i} \\ -F \int_0^T r'\left(s, X_s^x\right) \frac{\partial}{\partial x} X_s ds \end{array} \right] \\ &= \mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s, X_s^x) ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x \right) Y_{t_i} \\ -F \int_0^T r'\left(s, X_s^x\right) Y_s ds \end{array} \right] \end{aligned}$$

At this stage, equalling the two expressions of delta gives us:

$$\begin{aligned} & \mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s, X_s^x) ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x \right) \\ \int_0^T Y_{t_i} Y_t^{-1} \sigma(t, X_t) w(t)\, 1_{\{t \leq t_i\}} dt \\ -F \int_{s=0}^T \int_{t=0}^T r'\left(s, X_s^x\right) Y_s Y_t^{-1} \sigma(t, X_t) w(t)\, t 1_{\{t \leq s\}} dt ds \end{array} \right] \\ &= \mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s, X_s^x) ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x \right) Y_{t_i} \\ -F \int_0^T r'\left(s, X_s^x\right) Y_s\, ds \end{array} \right] \end{aligned}$$

Using the fact that this should hold for any f element of C_K^∞ and any function $r(.,.)$ also element of C_K^∞, we get that the following two quantities should be equal on any functions measurable,leading to conditions expressed with conditional expectations (where to simplify notations the x in superscript have been omitted):

$$\begin{aligned} & \mathbb{E}_x^Q\left[e^{-\int_0^T r(s,X_s^x)ds}\int_0^{t_i}\frac{Y_{t_i}\sigma(t,X_t)}{Y_t}w(t)\,dt|X_{t_1},...,X_{t_m}\right] \\ = \; & \mathbb{E}_x^Q\left[e^{-\int_0^T r(s,X_s^x)ds}Y_{t_i}|X_{t_1},...,X_{t_m}\right] \qquad \forall i=1...m \end{aligned} \tag{A.4}$$

$$\begin{aligned} & \mathbb{E}_x^Q\left[e^{-\int_0^T r(u,X_u^x)du}\int_{s=0}^T\int_{t=0}^T r'(s,X_s)\,Y_sY_t^{-1}\sigma(t,X_t)w(t)\,t1_{\{t\le s\}}dtds|X_{t_1},...,X_{t_m}\right] \\ = \; & \mathbb{E}_x^Q\left[e^{-\int_0^T r(u,X_u^x)du}\int_0^T r'(s,X_s)\,Y_sds|X_{t_1},...,X_{t_m}\right] \end{aligned} \tag{A.5}$$

this is exactly (M1) when the interest rate is a only function of the time □

Sufficient condition: If we know a function w that verifies two equations (A.4) and (A.5) and its Skorohod integral is L_2 integrable, the above proof can be conducted backwards:

$$\begin{aligned} delta &= \frac{\partial}{\partial x}\mathbb{E}_x^Q\left[\left(e^{-\int_0^T r(s,X_s)ds}f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\right)\right] \\ &= \mathbb{E}_x^Q\left[\begin{array}{c} \sum_{i=1}^m e^{-\int_0^T r(s,X_s)ds}\partial_i f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\frac{\partial}{\partial x}X_{t_i} \\ -\left(e^{-\int_0^T r(s,X_s)ds}f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\right)\int_0^T r'(s,X_s)\,\frac{\partial}{\partial x}X_sds \end{array}\right] \end{aligned}$$

then using the necessary conditions, we get

$$\begin{aligned} & \mathbb{E}_x^Q\left[\begin{array}{c} \sum_{i=1}^m e^{-\int_0^T r(s,X_s)ds}\partial_i f\left(X_{t_1},X_{t_2},...,X_{t_m}\right) \\ \int_0^T Y_{t_i}Y_t^{-1}\sigma(t,X_t)w(t)\,1_{\{t\le t_i\}}dt \\ -F\int_{s=0}^T\int_{t=0}^T r'(s,X_s)\,Y_sY_t^{-1}\sigma(t,X_t)\,w(t)\,t1_{\{t\le s\}}dtds \end{array}\right] \\ = \; & \mathbb{E}_x^Q\left[\begin{array}{c} e^{-\int_0^T r(s,X_s)ds}\sum_{i=1}^m \nabla_i f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\int_{t=0}^T D_tX_{t_i}w(t)\,dt \\ -F\int_{t=0}^T\int_{s=0}^T r'(s,X_s)\,D_tX_sw(t)\,dsdt \end{array}\right] \end{aligned}$$

which then using the expression of the Malliavin derivative in terms of the first variation process, leads to

$$\mathbb{E}_x^Q\left[\left\langle D_t\left(e^{-\int_0^T r(s,X_s)ds}f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\right),w(t)\right\rangle\right]$$

leading to the final result:

$$delta = \mathbb{E}_x^Q\left[F\delta(w)\right]$$

where in the last step, we made use of the integration by parts formula.□

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