Efficient Computation of Greeks for Discontinuous Payoffs by Transformation of the Payoff Function

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Abstract

Since price sensitivities are an important measure of risk, growing emphasis on risk management issues has suggested a greater need for their efficient computation. This paper explains how to get faster Greeks computation by means of the Malliavin weighting scheme as introduced by Fournié et al. (1999) and Benhamou (2000a). We examine different options, namely call, digital and corridor options in the case of a Black diffusion. Monte Carlo simulations confirm that the Malliavin method is a good variance reduction technique, all the more efficient that the option pay-off is discontinuous.

1 Introduction

The traditional approach of option pricing relies on hedging. Since the seminal work of Black Scholes (1973), the fair price of an option is given by the portfolio that replicates exactly the option payoff at maturity. If we introduce incompleteness in our model, the hypothesis of perfect replication should be relaxed. One should use different types of criteria to find a price. There is an extensive literature on super-replication or risk minimizing (see for example El Karoui and Quenez (1995), Jouini et al. (1996), Frey (1999)). However, this is not very realistic in many cases since it leads to too expensive prices. As a consequence, the derivatives industry still assumes a perfect replicating portfolio and is still very much concerned about the way of calculating it. This problem is commonly referred to as the computation of price sensitivities known as the Greeks.

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It is worth noticing that the numerical computation of the Greeks and the one of the option price are often two separate issues. One can get a good price estimation but poor results for the Greeks (Broadie and Glasserman (1997)) inconsistent with the model assumed[1]. The reason of this inefficiency lies in the way the Greeks are commonly computed. One estimates the Greeks by simply taking the finite difference estimator of two particular numerical results. Note that what we called finite difference should not be confused with finite difference method for PDEs. By finite difference, we mean that we calculate a discretized version of the Greeks. If we denote by $P(x)$ the option price with an initial underlying level of x, we know that there are three different ways of approximating the delta: forward difference $(P(x+\varepsilon) - P(x))/\varepsilon$, central difference $(P(x+\varepsilon) - P(x-\varepsilon))/2\varepsilon$, or backward difference scheme $(P(x) - P(x-\varepsilon))/\varepsilon^2$. Even if this method contains a natural antithetic method involved by the substraction between the terms $P(x+\varepsilon)$, $P(x)$, $P(x-\varepsilon)$, it embodies two different errors:

- discretization of the derivative function by a finite difference.
- imperfect estimation of the prices $P(x+\varepsilon)$, $P(x)$, and $P(x-\varepsilon)$.

For a discontinuous payoff function, it has been advocated that approximating the derivative function by a finite difference can lead to very unpleasant important errors (Glynn (1989)). There are indeed solutions to this problem. As pointed out first by Broadie and Glasserman (1997), one can avoid to differentiate the payoff. They showed that one can introduce a likelihood ratio. This method has the disadvantage to require an explicit formula for the density. Fournié et al. (1999) and later Benhamou (2000a) showed that in very general condition, one can transform the initial formula so as to shift the differential operator from the payoff function to the density function of the underlying. The result can be written as an expectation of the discounted payoff function denoted by $e^{-\int_0^T r_s ds} f(X_T)$ multiplied by a suitable weighting function *weight*:

$$Greek = \mathbb{E}^Q \left[e^{-\int_0^T r_s ds} f(X_T).\text{weight} \right] \tag{1.1}$$

In an inspiring Fournié et al. (1999) introduced the subject and gave particular cases for the weighting function. Benhamou (2000a) showed that their results were only particular cases of a more general theory. Introducing the weighting function generator, defined as the Skorohod integrand of the weight, He gave necessary and sufficient conditions for a function to serve as weighting function generator, characterizing all possible solutions. Moreover, Benhamou (2000a) and Fournié et al. (2000) showed independently that the optimal weight in the sense of the total variance should be measurable with respect to the variables involved in the payoff. As a

[1]We rule out here the issue of having a good price but wrong Greeks that could be implied by a wrong model as it is another subject and is not the motivation of this paper.

[2]Obviously, a finite difference method for PDE is precisely using a finite difference approximation of the different derivative to solve the PDE, hence the name.

consequence, the likelihood ratio was shown to be precisely the optimal weight. However, these were mainly theoretical studies, which argued that Malliavin weighted scheme should outperform standard methods. Two main points were missing. First, these authors never quantified precisely the comparative advantage. Second, they never specified which options were the most appropriate for the Malliavin weighted scheme. This is precisely to these questions that this paper tries to provide answers. In this paper, we examine the particular case of the Black diffusion. We try to quantify the variance reduction induced by the Malliavin method and to define an empirical typology of option for which the Malliavin based formula is more efficient than the traditional finite difference method.

The remainder of this paper is organized as follows. In section 2, we explain why the finite difference approximation used for Monte Carlo fails to get fast Greeks for discontinuous payoffs. This suggests to use Malliavin based formulae. In section 3, we give explicit formulae for options depending on a finite set of dates. In section 4, we examine simulation results that confirm our theoretical predictions: Malliavin formulae are more efficient for strongly discontinuous payoff options. We define a typology of option types for which the Malliavin based formula should be efficient and quantify the variance reduction on our numerical simulations. We briefly conclude in section 5, giving possible extensions. In the appendix section, we give a primer on Malliavin calculus and the important results for the Greeks as this is a new substantial subject and many readers may not be familiar with it.

2 Why a new method for the estimation of the Greeks?

In this section, after summarizing our model hypothesis, we explain why the finite difference, advocated to be quite fast by the use of common random numbers, fails to get efficient estimates of the Greeks in the case of discontinuous payoff options.

2.1 Assumptions: Black model

We consider a continuous time trading economy with a limited period of horizon $T \in [0, T_\infty]$ ($T_\infty < +\infty$). The uncertainty is characterized by a complete probability space (- , F, Q) where - is the state space, F is the σ-algebra representing the measurable events, and Q is the risk neutral probability measure[3]. The information evolves according to the augmented right continuous complete filtration $\{F_t, t \in [0, T_\infty]\}$ generated by a standard one dimensional Brownian Motion $\{W_t, t \in [0, T_\infty]\}$. We assume the underlying price process $(X_t)_{t\in[0,T]}$ follows a geometric Brownian motion with a time dependent volatility, given by the Ito process solution of the following

[3] since this basic model assumes markets completeness, this risk neutral probability measure exits and is uniquely defined

Stochastic Differential Equation (2.1)

$$dX_t = r_t X_t dt + \sigma_t X_t dW_t \tag{2.1}$$

with initial condition $X_0 = x$, and with r_t the deterministic risk free interest rate and σ_t the deterministic Black (1976) volatility. The instantaneous variance $\sigma(t, X_t)$ of the process increment dX_t is given by $\sigma_t X_t$ and its drift $b(X_t)$ by $r_t X_t$. The "first variation" process, a Ito process itself, derivative of the Ito Process $(X_t)_{t\in[0,T]}$ with respect to its initial condition x, is proportional to the underlying process

$$X_t = xY_t$$

The proportionality between the first variation process and the underlying implies a proportionality between vega and gamma (see Benhamou (2000c)). Therefore, the gamma can be obtained easily for a known vega and vice versa.

2.2 Inefficiency of simulations for discontinuous payoff when using finite difference approximation for the Greeks

As pointed out by Glynn (1989), by Glasserman and Yao (1992), Boyle, Broadie and Glasserman (1997) and by L'Ecuyer and Perron (1994), a finite difference scheme[4] can be improved by taking common random numbers for the computation of the Greeks. If we denote by $P(x)$ the option price with an initial underlying's level of x, by $X_T(\varepsilon)$ the underlying's value at time T with an initial condition $x+\varepsilon$, and by K the strike of the option, a finite difference scheme for the particular case of the delta leads to approximate the delta by a finite difference approximation like $(P(x+\varepsilon) - P(x))/\varepsilon$. The decisive element for the variance of this estimator is the variance of the numerator, which turns out to be equal to

$$Var(P(x)) + Var(P(x+\varepsilon)) - 2Cov(P(x+\varepsilon), P(x)) \tag{2.2}$$

From the equation above, one can see that the estimator will have a lower variance when the two prices $P(x+\varepsilon)$ and $P(x)$ are positively correlated. This is why using common random numbers for the simulation of the two options prices: $P(x+\varepsilon)$ and $P(x)$, is efficient. Going even further, L'Ecuyer and Perron (1994) proved that the convergence rate is $n^{-1/2}$, which is the best that can be obtained from Monte Carlo simulations. The efficiency of simulations with common numbers relies on the fast mean-square convergence of $P(x+\varepsilon)$ to $P(x)$. The rate of $n^{-1/2}$ unfortunately does not apply in all cases. For example, it fails to hold in the case of the digital call, an option paying 1 in the case of an underlying above the strike $X_T > K$ and zero elsewhere. This comes from the slow mean square convergence of $P(x+\varepsilon)$ to $P(x)$. The difference between

[4] Let us again insist that by finite difference scheme, we do not mean finite difference method for PDEs but the discretized form of the derivative function (i.e. for the delta the centered finite difference scheme is $(P(x+\varepsilon) - P(x-\varepsilon))/2\varepsilon$)

the shifted digital call $P(x+\varepsilon)$ and the regular digital call $P(x)$ is given by a probability times a discount rate squared:

$$\mathbb{E}\left[|P(x+\varepsilon)-P(x)|^2\right]=e^{-2rT}P\left[X_T<K<X_T(\varepsilon)\right]$$

Assuming an homogeneous underlying process, $X_T(\varepsilon)=X_T*\left(1+\frac{\varepsilon}{x}\right)$, it leads to a convergence rate of ε for this probability. Writing with the Landau symbol O[5], we get that the convergence of $P(x+\varepsilon)$ to $P(x)$ is only linear in ε :

$$\mathbb{E}\left[|P(X_0+\varepsilon)-P(X_0)|^2\right]=O(\varepsilon)$$

On the contrary, in the case of the plain vanilla call option, it can be shown (see for example Broadie and Glasserman (1996)) that for the geometric Brownian motion, the convergence rate is of ε^2

$$\begin{aligned}\mathbb{E}\left[|P(x+\varepsilon)-P(x)|^2\right] &\leq \mathbb{E}\left[|X_T(\varepsilon)-X_t|^2\right]\\ &\leq \varepsilon^2\mathbb{E}\left[e^{(r-\mu)T+\sigma\sqrt{T}Z}\right]\end{aligned}$$

where Z is a normal variable $N(0,1)$, leading to

$$\mathbb{E}\left[|P(x+\varepsilon)-P(x)|^2\right]=O\left(\varepsilon^2\right)$$

This is why the methodology of finite difference under-performs for all discontinuous type options like simple digital, corridor (option which pays 1 if the underlying at time T is inside an interval $L<X_T<H$), barrier option and so on.

2.3 Introduction to a new method: the Malliavin weighted scheme

To overcome this problem, Fournié et al. (1999) and Benhamou (2000a) advocated the use of an integration by parts formula so as to construct smooth estimators of the Greeks. The idea is to avoid differentiating the payoff function and instead to differentiate the density function. Usually, the density function is very smooth and can be easily differentiated (in the case of the geometric Brownian motion, the density function is a C^∞ function proportional to $e^{-\frac{z^2}{2}}$). More precisely, in the case of the Black Scholes model, writing the delta as an explicit integral, we can do an integration by parts

$$\begin{aligned}delta &= \frac{\partial}{\partial x}\int_{\mathbb{R}} e^{-rT}\left(xe^{(r-\mu)T+\sigma\sqrt{T}z}-K\right)^+\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\\ &= e^{-rT}\int_{\mathbb{R}}\frac{1}{x\sigma\sqrt{T}}\frac{\partial}{\partial z}\left(xe^{(r-\mu)T+\sigma\sqrt{T}z}-K\right)^+\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\end{aligned}$$

[5] Big O notation also called Landau's symbol, is a symbolism used in mathematics to describe the asymptotical behavior of functions. Basically, it tells you how fast a function grows or declines. $f(x)=O(g(x))$ if and only if there exist constants N and C such that $|f(x)|\leq Cg(x)$ for all $x>N$. Intuitively, this means that f does not grow faster than g. (see for example Hardy and Wright (1979))

which leads to a simple expression of the delta:

$$= \quad e^{-rT}[\frac{1}{x\sigma\sqrt{T}} \left(xe^{(r-\mu)T+\sigma\sqrt{T}z} - K\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}]_{-\infty}^{+\infty}$$
$$+e^{-rT} \int_{\mathbb{R}} \frac{1}{x\sigma\sqrt{T}} \left(xe^{(r-\mu)T+\sigma\sqrt{T}z} - K\right)^+ \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

when re-expressing this as an expectation, we get:

$$delta = \mathbb{E}(\frac{e^{-rT}}{x\sigma T} W_T \ (X_T - K)^+)$$

where W_T stands for the Brownian motion at time T.

2.4 How to derive necessary and sufficient conditions for the Malliavin weight?

The goal of this section is to introduce intuitively the weighting function generator and see how one can use Malliavin calculus to do the integration by parts that we did explicitly in the previous section (for a rigorous presentation of the weighting function generator, one can refer to Benhamou (2000a)).We will assume that the underlying price follows the following jump diffusion

$$X : dX_t = b\,(t, X_t)\, dt + \sigma\,(t, X_t)\, dW_t + \lambda\,(t)\, dJ_t \tag{2.3}$$

its first variation process (also called the tangential process) $(Y_t)_{t\geq 0}$ is defined as , $Y_t = \frac{\partial}{\partial x} X_t$,. obviously $Y_0 = 1$ and

$$Y : dY_t = \frac{\partial}{\partial x} b\,(t, X_t)\, Y_t dt + \frac{\partial}{\partial x} \sigma\,(t, X_t)\, Y_t dW_t$$

We will assume that the option depends on a set of discrete observation time $X_{T_1}, ..., X_{T_n}$ with $T_n = T$. We will show now how to characterize the weight of the formula (1.1). In fact, what we will show here is even more general.

Proposition 1 *If there exists a formula of the type*

$$Delta = \mathbb{E}^Q \left[e^{-\int_0^T r_s ds} f(X_{T_1}, ..., X_{T_n}).weight \right]$$

with the weight expressed as a Skorohod integral

$$weigth = \delta\,(u)$$

where the Skorohod integral is L^2 integrable, then the integrand that we called in the rest of the paper, weighting function generator (in line with the concept first introduced by Benhamou (2000a)) has to satisfy some necessary and sufficient conditions:

$$\mathbb{E}^Q \left[Y_{T_i} | X_{T_1}, ..., X_{T_n} \right] = \mathbb{E}^Q \left[\int_0^{T_i} \sigma\,(s, X_s)\, Y_{T_i} Y_s^{-1} u ds | X_{T_1}, ..., X_{T_n} \right] \tag{2.4}$$

Proof: The proof consists in using the integration by parts formula of the Malliavin calculus. We will always assume that the functions are smooth enough to justify the derivation as well as the interchange of the integration and the derivation operator. (for technical details, the reader can refer to Benhamou (2000a)). On one hand, we can say that the *Delta* is given by the standard derivation with respect to the initial underlying price

$$Delta = \frac{\partial}{\partial x}\mathbb{E}^Q\left[e^{-\int_0^T r_s ds} f(X_{T_1},....,X_{T_n})\right] = \mathbb{E}^Q\left[e^{-\int_0^T r_s ds}\sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_{T_1},....,X_{T_n})Y_{T_i}\right] \quad (2.5)$$

where we have implicitly interchanged the derivation and integration operator.

On the other hand, we should have a formula of the type $Delta = \mathbb{E}^Q\left[e^{-\int_0^T r_s ds} f(X_{T_1},....,X_{T_n})\delta(u)\right]$, which can be transformed by the integration by parts formula (B.3) into

$$Delta = \mathbb{E}^Q\left[e^{-\int_0^T r_s ds}\int_0^T D_s f(X_{T_1},....,X_{T_n})u ds\right]$$

which can in term be transformed using the properties of Malliavin derivatives (B.2) into

$$\begin{aligned} Delta &= \mathbb{E}^Q\left[e^{-\int_0^T r_s ds}\sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_{T_1},....,X_{T_n})\int_0^T D_s X_{T_i} u ds\right] \\ &= \mathbb{E}^Q\left[e^{-\int_0^T r_s ds}\sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_{T_1},....,X_{T_n})\int_0^T \sigma(s,X_s) Y_{T_i} Y_s^{-1} 1_{\{s\leq T_i\}} u ds\right] \quad (2.6) \end{aligned}$$

where we have used in the last equation the expression of the Malliavin derivative with respect to the first variation process (B.3). The two equation members (2.5) and (2.6) can be equivalent if and only this

$$\mathbb{E}^Q\left[e^{-\int_0^T r_s ds}\sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_{T_1},....,X_{T_n})Y_{T_i}\right] = \mathbb{E}^Q\left[e^{-\int_0^T r_s ds}\sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_{T_1},....,X_{T_n})\int_0^{T_i} \sigma(s,X_s) Y_{T_i} Y_s^{-1} u ds\right]$$

and this should hold for any function f of $X_{T_1},....,X_{T_n}$, which is equivalent of the equality of conditional expectations: for each $i = 1..n$

$$\mathbb{E}^Q\left[Y_{T_i}|X_{T_1},....,X_{T_n}\right] = \mathbb{E}^Q\left[\int_0^{T_i} \sigma(s,X_s) Y_{T_i} Y_s^{-1} u ds | X_{T_1},....,X_{T_n}\right]$$

□

Remark 1 *We can notice that a sufficient condition is given by*

$$\int_0^{T_i} \sigma(s,X_s) Y_s^{-1} u ds = 1$$

for any $i = 1..n$. In the case of Black Scholes, this leads to have $\sigma(s,X_s) = \sigma X_s$, while $Y_s = X_s/x$ so that we have

$$\int_0^T x\sigma_t w^{delta}(t) 1_{\{t\leq t_i\}} dt = 1 \quad \forall i = 1...m$$

2.5 Importance sampling and Malliavin weights

The results obtained state that any Greeks can be written as an expectation of the discounted payoff function multiplied by a suitable weighting function *weight* (equation (1.1)). This remind us the importance sampling method, where one changes the probability measure in order to have the different paths reaching the target. In fact, the Malliavin weight is just a powerful way of doing some importance sampling but without requiring an explicit knowledge of the Radon-Nikodyn derivatives and the change of measure to do. Let us more precisely examined the case of the delta. The option price is computed as the expectation of the discounted payoff under the risk-neutral payoff.

$$P(x_0) = \mathbb{E}^Q\left[e^{-\int_0^T r_s ds} f(X_T(x_0))\right]$$

, where we have denoted the underlying price $X_T(x_0)$, function of the initial underlying price x_0. When we bump the initial price x_0 to $x_0 + \varepsilon$, it is intuitively as if we were changing the measure. In the simple case of the European option, we are only concerned about the value of the underlying price at the final time $T > 0$, $X_T(x_0 + \varepsilon)$. Interestingly, we can see that the bumped process $X_T(x_0 + \varepsilon)$ could be thought as the unbumped process but with a modified drift. In the case of the Black Scholes model, the bumped process is related to the unbumped process by the following change of drift

$$\begin{aligned} X_T(x_0+\varepsilon) &= (X_0+\varepsilon)\exp\left(\sigma W_T + \left(r - \frac{\sigma^2}{2}\right)T\right) \\ &= X_0 \exp\left(\sigma W_T + \left(\widetilde{r} - \frac{\sigma^2}{2}\right)T\right) \end{aligned}$$

with $\widetilde{r} = r + \frac{1}{T}\ln\left(\frac{(X_0+\varepsilon)}{X_0}\right)$. Computing a price with the bumped process is therefore equivalent to computing a price with an un-bumped process but under a different measure that we denote $\widetilde{Q}(\varepsilon)$.

$$P(x_0+\varepsilon) = \mathbb{E}^Q\left[e^{-\int_0^T r_s ds} f(X_T(x_0+\varepsilon))\right] = \mathbb{E}^{\widetilde{Q}(\varepsilon)}\left[e^{-\int_0^T r_s ds} f(X_T(x_0))\right]$$

As we saw in the section on Malliavin weight, the change of measure is independent from the payoff. It is only depending on the observation dates used for the payoff. The variation of prices is consequently equal to the difference of two expectations with two different measures but with the same underlying process:

$$\text{Variation of price} = \mathbb{E}^{\widetilde{Q}(\varepsilon)}\left[e^{-\int_0^T r_s ds} f(X_T(x_0))\right] - \mathbb{E}^Q\left[e^{-\int_0^T r_s ds} f(X_T(x_0))\right]$$

but we know that we can reexpress this under the original risk measure by using the Radon Nikodym derivative $\frac{d\widetilde{Q}(\varepsilon)}{dQ}$ so that we have in fact that the variation of price is equal to

$$\text{Variation of price} = \mathbb{E}^Q\left[e^{-\int_0^T r_s ds} f(X_T(x_0))\left(\frac{d\widetilde{Q}(\varepsilon)}{dQ} - 1\right)\right]$$

The delta which is purely the limit of the variation prices divided by the bump size can be reexpressed as

$$\begin{aligned}\text{delta} &= \lim_{\varepsilon\to 0}\frac{\text{Variation of price}}{\varepsilon}\\ &= \mathbb{E}^{Q}\left[e^{-\int_0^T r_s ds} f\left(X_T\left(x_0\right)\right)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\frac{d\widetilde{Q}\left(\varepsilon\right)}{dQ}-1\right)\right]\end{aligned}$$

or rexpressed in a similar way as the equation 1.1, we have obtained that the weight is equal to

$$weight = \frac{\partial}{\partial\varepsilon}\frac{d\widetilde{Q}\left(\varepsilon\right)}{dQ}$$

Obviously, there exists an infinity of measures that enable to do the change of drift from the bumped to the unbumped process, explaining intuitively why there exists an infinity of Malliavin weight. The power of the Malliavin method is to be able to calculate the weight without requiring an explicit knowledge of the change of measure. In a sense, Malliavin calculus is a smart way of doing importance sampling since one does not require to give an explicit expression of the change of measure.

3 Determination of the Malliavin Weights

This section is a straightforward application of the results derived by Benhamou (2000a). Those were theoretical formulae in a very general framework. We show how to apply them to the particular case of a Black diffusion for options on discrete observations (for the Asian option, see Benhamou (2000b)). We remind important characteristics of Malliavin[6] weights:

- all Greeks can be written as the expected value of the payoff times a weight function.
- the weight functions are independent from the payoff function. The method efficiency is therefore increased for discontinuous payoff options.
- the weight functions are given as the Skorohod integral[7] of some generator, characterized by necessary and sufficient conditions being expressed through conditional expectations. However, since it is easier to handle and still very robust, we use sufficient and stronger conditions that are the equality almost surely of the terms inside the conditional expectations.
- there is an infinity of solutions for the generator function. However, it is more efficient to choose weight functions expressed with the same points of the Brownian motion trajectory as the option payoff. For an option depending on a series of dates $t_1 < t_2 < ... < t_m$ it is appropriate to choose a weight function expressed in terms of $W_{t_1}, ..., W_{t_n}$. No extra

[6] See the appendix section for a primer on Malliavin calculus

[7] See the appendix section for a brief introduction to the Skorohod integral. Note also that for adaptive process, the Skorohod integral coincides with the Ito integral.

simulation is required for the computation of the weight function and it can be shown that the variance of the weight function is minimum.

The latter stronger condition can in some cases not be fulfilled. In these cases, it becomes very difficult to determine the most efficient weight functions. In the rest of the paper, we take the convention that $t_0 = 0$. We denote by F the discounted payoff. The option is depending on a series of increasing dates $t_1 < t_2 < ... < t_m$ with $t_m = T$. This dependence is very general. It can represent many options depending on a finite set of dates, as in discrete Asian, barrier and lookback options.

3.1 Delta

We have seen in the section (2.4) that the delta is equal to the expected value of the discounted payoff F times a weighting function expressed as a Skorohod integral (equation (3.1))

$$\text{delta} = E_x^Q \left[F\delta \left(w^{\text{delta}} \right) \right] \tag{3.1}$$

where the function w^{delta}, called the weighting function generator, has to satisfy a sufficient condition given by equation (3.2)

$$\int_0^T x\sigma_t w^{\text{delta}}(t)\, 1_{\{t \leq t_i\}} dt = 1 \quad \forall i = 1...m \tag{3.2}$$

as well as the L^2 integrability of its Skorohod integral, which is the condition for the existence of the Skorohod integral (Oksendal (1997) page 22). Moreover, if the solution for the weighting function generator is an adapted process of $L^2(- \times [0,T])$, the Skorohod integral reduces to the Ito integral. This is because the Skorohod integral coincides with the Ito integral on the space of adapted process of $L^2(- \times [0,T])$.

Among the different weighting function generators, it can be shown that the one with the lowest variance is the one expressed in terms of the same points of the Brownian motion as the option payoff, that is to say $W_{t_1}, ..., W_{t_n}$. This implies to use a piecewise constant generator. We denote by $(\lambda_i)_{i=1..n}$ the sequence initialized with $\lambda_1 = \frac{1}{x\int_0^{t_1} \sigma_t dt}$ and defined by the following recurrence: for $1 \leq i_0 < n$, λ_{i_0+1} is given by

$$\lambda_{i_0+1} = \frac{\frac{1}{x} - \sum_{i=1}^{i_0} \int_{t_{i-1}}^{t_i} \lambda_i \sigma_t dt}{\int_{t_{i_0}}^{t_{i_0+1}} \sigma_t dt}$$

With these definitions, the most appropriate solution for our generator (in term of computation) is given by the following proposition:

Proposition 2 *The piecewise solution for our generator is given by*

$$w^{delta}(t) = \sum_{i=1}^{n} \lambda_i 1_{[t_{i-1}, t_i[}(t) \tag{3.3}$$

leading to the following expression for the delta

$$delta = \mathbb{E}_x^Q \left[F \sum_{i=1}^{n} \lambda_i \left[W_{t_i} - W_{t_{i-1}} \right] \right]$$

Proof: the solution (3.3) verifies condition (3.2).□

Remark 2 *In the particular case of an option depending only on a final date T (European option) with a Black Scholes diffusion ($\sigma_t = cte = \sigma$), we find the following particular solution:*

$$\delta = \mathbb{E}\left[e^{-\int_0^T r_s ds} f\left(X_T\right) \frac{W_T}{T\sigma x} \right] \tag{3.4}$$

The weight function is very simple in this special case. It is the Brownian motion divided by the maturity of the option times the volatility times the initial condition. This suggests that for an option close to maturity, the Malliavin weight of the delta should explode. Indeed, when the option is close to maturity, the condition (3.2) leads to increase the generator. The problem of a wider hedge close to the maturity is well-known, especially in the literature about barrier options. As far as the volatility is concerned, the intuition is that more volatility makes the option price more convex. It smoothens in a way the Greeks. This is why it is consistent with the decrease of the Malliavin weight with respect to the volatility parameter. The gamma (Γ) computation is harder than for the other Greeks since it is a second order derivative. However, since in the Black model, the first variation and underlying process are proportional, there is a proportionality between gamma and vega (see Benhamou (2000c)). The vega v is given by (in the case of a European option)

$$v = x^2 \sigma T * \gamma$$

Using this property enables us to compute easily the gamma. That is why we do not develop any further our analysis for the gamma and refer to the vega section for an elegant way of calculation. So as to be as extensive as possible, we can mention that a straightforward computation of the gamma can be done. Using the theoretical results on the Malliavin weight function (see Benhamou (2000a)), we find that one particular solution for the weight function of the gamma is given by:

$$weight_\Gamma = \left[\left(\int_0^T \frac{\lambda_t}{x\sigma_t} dW_t \right)^2 - \int_0^T \left(\frac{\lambda_t}{x\sigma_t} \right)^2 ds - \int_0^T \frac{\lambda_t}{x^2 \sigma_t} dW_t \right] \tag{3.5}$$

Remark 3 *In the particular case of an option depending only on a final date T (European option) with a Black Scholes diffusion ($\sigma_t = cte = \sigma$), we find:*

$$\Gamma = E\left[e^{-\int_0^T r_s ds} f\left(X_T\right) \frac{1}{T\sigma x^2} \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right]$$

3.2 Rho

The meaning of the classical rho is to examine the price sensitivity with respect to the risk-free rate. The results derived by Benhamou (2000a) are for a perturbation on the drift part of the diffusion of the underlying. However, a change in the risk-free rate impacts in two ways. It alters the drift of the underlying diffusion but it also modifies the discount factor:

$$\begin{aligned} rho &= \frac{\partial}{\partial\varepsilon}\mathbb{E}_x^Q\left[e^{-\int_0^{t_m} r_s ds} f\left(X_{t_1}^{\varepsilon,rho},...,X_{t_m}^{\varepsilon,rho}\right)\right] \\ &\quad + \frac{\partial}{\partial\varepsilon}\mathbb{E}_x^Q\left[e^{-\int_0^{t_m} r_s+\varepsilon\widetilde{r}_s ds} f\left(X_{t_1},...,X_{t_m}\right)\right] \end{aligned} \tag{3.6}$$

where $X_t^{\varepsilon,rho}$ stands for the underlying with a perturbed drift $\widetilde{b}(u,X_u) = X_u\widetilde{r}_u$, and where the limit is almost surely, taken for $\varepsilon = 0$. The second term can be calculated by interchanging the expectation and the derivative operator and differentiating the discount factor with respect to ε:

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}_x^Q\left[e^{-\int_0^{t_m} r_s+\varepsilon\widetilde{r}_s ds} f\left(X_{t_1},...,X_{t_m}\right)\right] = \mathbb{E}_x^Q\left[-\int_0^{t_m}\widetilde{r}_s ds\; e^{-\int_0^{t_m} r_s} f\left(X_{t_1},...,X_{t_m}\right)\right]$$

Like in the case of the delta, the first term of the right hand side of equation (3.6) can be expressed as the expected value of the discounted payoff F times a weight function expressed as a Skorohod integral

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}_x^Q\left[e^{-\int_0^{t_m} r_s ds} f\left(X_{t_1}^{\varepsilon},...,X_{t_m}^{\varepsilon}\right)\right] = \mathbb{E}_x^Q\left[F\delta\left(w^{\text{rho}}\right)\right] \tag{3.7}$$

where the function w^{rho}, called the weight function generator has to satisfy a sufficient condition given by the following equation

$$\int_0^{t_i} x\sigma_t w^{\text{rho}}(t)\,dt = \int_0^{t_i} x\widetilde{r}_t dt \quad \forall i = 1...m \tag{3.8}$$

as well as the L^2 integrability of its Skorohod integral. An obvious solution is $w^{\text{rho}} = \frac{\widetilde{r}_t}{\sigma_t}$. Using the fact that the classical rho is the price sensitivity with respect to the risk-free rate, we get the following proposition:

Proposition 3 *The rho is given by:*

$$rho = E^Q\left[F * \left(\int_0^{t_m}\frac{\widetilde{r}_t}{\sigma_t}dW_t - \int_0^{t_m}\widetilde{r}_t dt\right)\right]$$

3.3 Vega

The vega is the perturbation along the volatility term of the diffusion. We write the perturbation as $\widetilde{\sigma}(t,X_t) = \widetilde{\sigma}_t X_t$. Like in the case of the other Greeks, we can define a weight function characterized by its generator. The weight function generator $w^{vega}(.)$ should satisfy:

$$\int_{t=0}^{t_i}\sigma_t w^{vega}(t)\,dt = \int_{t=0}^{t_i}\widetilde{\sigma}_t dW_t - \int_{t=0}^{t_i}\sigma_t\widetilde{\sigma}_t dt \quad \forall i = 1...n \tag{3.9}$$

as well as the L^2 integrability of its Skorohod integral, which is the existence of the Skorohod integral. One possible solution is a piecewise constant solution. We denote by $(\lambda_i)_{i=1..n}$ the sequence initialized with

$$\lambda_1 = \frac{\int_{t=0}^{t_1}\widetilde{\sigma}_t dW_t - \int_{t=0}^{t_1}\sigma_t\widetilde{\sigma}_t dt}{\int_{t=0}^{t_1}\sigma_t dt}$$

and defined by the following recurrence: for $1 \leq i_0 < n$, λ_{i_0+1} is given by

$$\lambda_{i_0} = \frac{\int_{t=0}^{t_{i_0}} \widetilde{\sigma}_t dW_t - \int_{t=0}^{t_{i_0}} \sigma_t \widetilde{\sigma}_t dt - \sum_{i=1}^{i_0-1} \int_{t_{i-1}}^{t_i} \lambda_i \sigma_t dt}{\int_{t_{i_0-1}}^{t_{i_0}} \sigma_t dt}$$

We then get that there is a piecewise constant solution as the following proposition states it:

Proposition 4 *piecewise constant solution*

One particular solution for the weight function generator is defined by:

$$w^{vega}(t) = \sum_{i=1}^{n} \lambda_i 1_{[t_{i-1},t_i[}(t) \tag{3.11}$$

Proof: The solution given by equation (3.11) verifies the necessary and sufficient condition to be a weighting function generator equation (3.9).□

Corollary 1 *In the case of an option depending only on a final date denoted by T, we get*

$$vega = \mathbb{E}\left[e^{-\int_0^T r_s ds} f(X_T)\, \delta\left(\frac{\int_0^T \widetilde{\sigma}_t dW_t - \int_0^T \sigma_t \widetilde{\sigma}_t dt}{\int_{t=0}^T \sigma_t dt}\right)\right]$$

Proof: immediate, since the weight function is defined as the Skorohod integral of the particular solution for the weighting function generator w^{vega} given by the equation (3.11).□

Corollary 2 *In the Black Scholes case, we get*

$$vega = \mathbb{E}\left[e^{-\int_0^T r_s ds} f(X_T) \frac{\widetilde{\sigma}}{\sigma T}\left(W_T^2 - T - \sigma T W_T\right)\right] \tag{3.12}$$

Proof: Using the fact that the Skorohod integral is a linear operator and that the Skorohod integral reduces to the Ito integral for adapted process, we get

$$\delta\left(\frac{\widetilde{\sigma} W_T}{\sigma T} - \widetilde{\sigma}\right) = \frac{\widetilde{\sigma}}{\sigma T} \int_{u=0}^T \int_{v=0}^T dW_u dW_v - \widetilde{\sigma} W_T$$

We need to calculate $\int_{u=0}^T \int_{v=0}^T dW_u dW_v$. This expression can be seen as a Wiener Chaos term of second order and is related to the Hermite polynomial of second order, so that (Øksendal (1997) page 19)

$$\int_{u=0}^T \int_{v=0}^T dW_u dW_v = W_T^2 - T$$

Putting all these terms together leads to the result.□

Corollary 3 *The classical vega is given by*

$$Classical\ vega = E\left[e^{-\int_0^T r_s ds} f(X_T)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right]$$

Proof: to obtain the classical vega, we must divide the above formula (3.12) by $\widetilde{\sigma}$.□

4 Numerical Result on the Efficiency of Malliavin weights

In this section, we compare the results of Malliavin weighted simulations with the ones obtained by a centered finite difference approximation $(P(x+\epsilon)-P(x-\epsilon))/2\epsilon$ for three different types of options in the Black-Scholes framework (so as to have closed formula):

- a corridor option: the payoff, given by $1_{\{S_{\max >} S_T > S_{\min}\}}$, displays two discontinuities. This is exactly the type of options we are targeting to since it has two discontinuities
- a binary call : the payoff given by $1_{\{S_T > S_{\min}\}}$ displays only one discontinuity. The payoff is smoother than the one of a corridor option.
- a vanilla call. This last example is to examine the impact of the formula when there is a smooth payoff.

A point we had to resolve at first, was the type of simulations to use. Boyle Broadie and Glasserman (1997), Caflisch Mokoroff and Owen (1997), Galanti and Jung (1997), Boyle Joy and Tan (1997), Papageorgiou and Traub (1996), Paskov (1994), Paskov and Traub (1995) and Williard (1997) show that low-discrepancy sequences are more efficient than random sequences for low dimension problems. Bratley, Fox and Niederreiter (1992), Galanti and Jung (1997), Morokoff (1997) and Moskovitz and Caflish (1995) demonstrate that low discrepancy sequences become less efficient for high dimensions. Galanti and Jung (1997) demonstrate that the Sobol sequence exhibits better convergence properties than either the Halton and Fauré sequences. Therefore, we used the Sobol sequence. We used common random numbers: the perturbed and unperturbed Wiener paths in the finite difference simulation were the same. Since the Sobol sequence fills the space with a pseudo periodicity, the simulations display pseudo-periodicity as well. We took the same parameters in the three option examples: X_0=100, r=5%, σ=15%, T=1 year, $S_{\min}$=95, $S_{\max}$=105, K=100. We display for each option the delta and the gamma. Rho and vega parameters lead to same results and are given for illustrative purpose in the particular case of the corridor option.

The results consist in two remarks:

- For discontinuous payoff function, as is the case of digital and corridor option, with a mean-square convergence of the shifted option $P(X_0+\varepsilon)$ to $P(X_0)$ linear in ε (see section 2.2, page 4), Malliavin formula outperforms finite difference method. This is because Malliavin simulation has lower simulation variance and converges faster. This comes from two self-reinforcing facts. First, Malliavin technique uses a smoothened payoff. Second, finite difference method is lengthy because of the slow mean-square convergence of the shifted option $P(X_0+\varepsilon)$ to the normal one $P(X_0)$.

- On the contrary, for vanilla options, smooth enough not to require the integration by parts technique and for which the mean-square convergence of a shifted option $P(X_0 + \varepsilon)$ to the normal one $P(X_0)$ is quadratic in ε (see section 2.2, page 4), finite difference outperforms the Malliavin-based method.

4.1 Comparative analysis: Finite Difference versus Malliavin weighted scheme

4.1.1 Corridor Option

This important example illustrates the drastic efficiency of the Malliavin theory. The corridor option pays 1 if the underlying at maturity is inside the corridor: payoff equal to $1_{\{S_{\max} > S_T > S_{\min}\}}$. The out-performance of the Malliavin simulation is illustrated by the figures 1, 2 which display the delta and gamma of the corridor option. Results on the vega and rho are similar to the one of the delta and gamma and are given in the appendix section as figure 7 and 8. A more quantitative analysis of the result is given in the section 4.2.

The figure 1 compares the two methods: Malliavin weighted scheme (black line) and the finite difference method (grey line). The Malliavin weighted scheme converges to the right answer fast with almost no oscillations, whereas the finite difference estimator fluctuates with a pseudo periodicity around the correct value

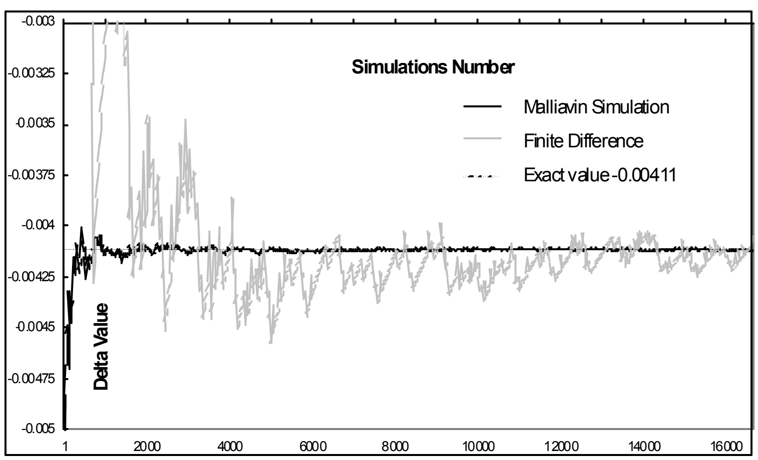

Figure 1: Efficiency of the Malliavin weighted scheme for the computation of the delta of a Corridor option

The figure 2 examines the computation of the gamma. Like in the case of the delta, the Malliavin weighted scheme out-performs dramatically compared to the finite difference method. It is worth noticing that this out-performance is even more pronounced for the gamma than for the delta.

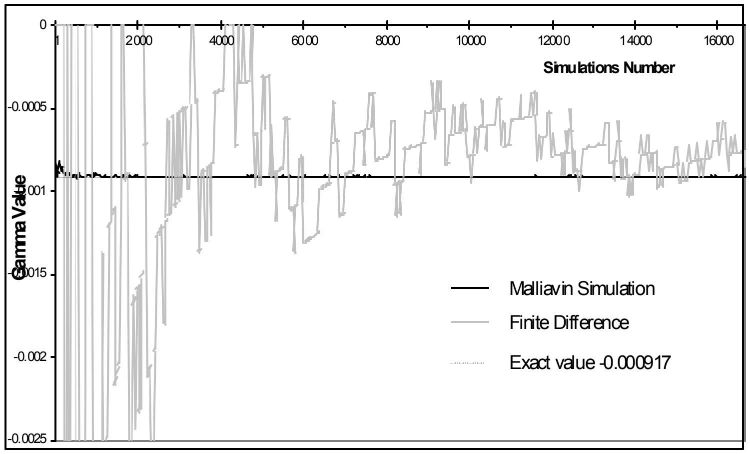

Figure 2: Efficiency of the Malliavin weighted for the computation of the gamma of a Corridor option

4.1.2 Binary Option

The binary option is a didactic example of a payoff function with small discontinuity $(1_{S_T>S_{\min}})$. Like for the corridor option, Malliavin simulation computes faster and more accurately the Greeks than the finite difference method. Finite difference simulation performs poorly since the mean-square convergence of the shifted option $P(X_0+\varepsilon)$ to $P(X_0)$ is only linear in ε. Figures 3 and 4 are respectively example of delta and gamma computation. They illustrate the outperformance of the Malliavin weighted scheme.

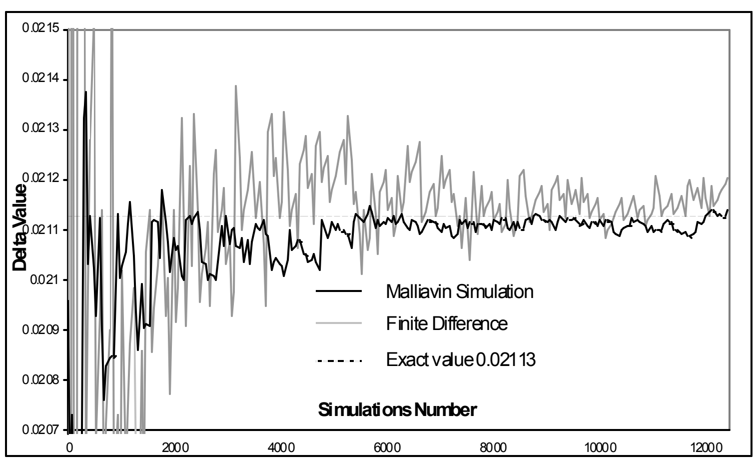

Figure 3: Comparison of the computation of the Delta of a Binary option by finite differences and by Malliavin weighted scheme

Like for the corridor option, Malliavin outperformance is more pronounced for the gamma than the delta as a comparative study of figure 3 and 4 shows. Gamma is a second order Greek. This suggests an increased efficiency for higher order Greeks.

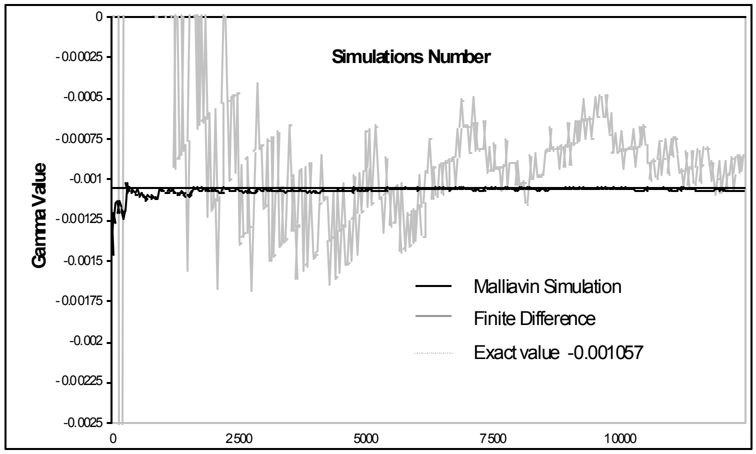

Figure 4: Comparison of the computation of the Gamma of a Binary option by finite differences and by Malliavin weighted scheme

4.1.3 Call Option

Last but not least, the Call option is an instructive case of a smooth payoff function. Since the payoff function does not present any strong discontinuity, it is smooth enough not to require any integration by parts smoothing. Therefore, the Malliavin simulation does not provide any technical advantage. Indeed, since the mean-square convergence of the shifted option $P(X_0+\varepsilon)$ to the normal one $P(X_0)$ is quadratic in ε, the finite difference method embodies a pseudo antithetic variance reduction. As a consequence, it converges faster than the Malliavin method as shown by figures 5 and 6, which represent respectively the delta and gamma.

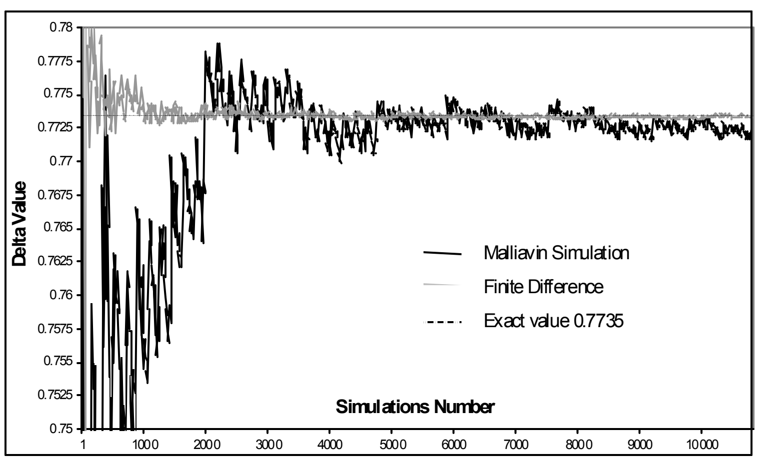

Figure 5: Comparison of the computation of the Delta of a Binary option by finite differences and by Malliavin weighted scheme

Even for the case of the gamma, which is a second order derivative, the finite difference is more efficient than the Malliavin based formula as proved by figure 6. The second order smoothing is not enough to offset the quadratic convergence of the shifted option $C(X_0+\varepsilon)$ to the normal one $C(X_0)$ as shown by figure 6. However, the comparative study of the figures 5 and 6 still indicates an increased efficiency of the Malliavin weighted scheme for second order Greeks like gamma.

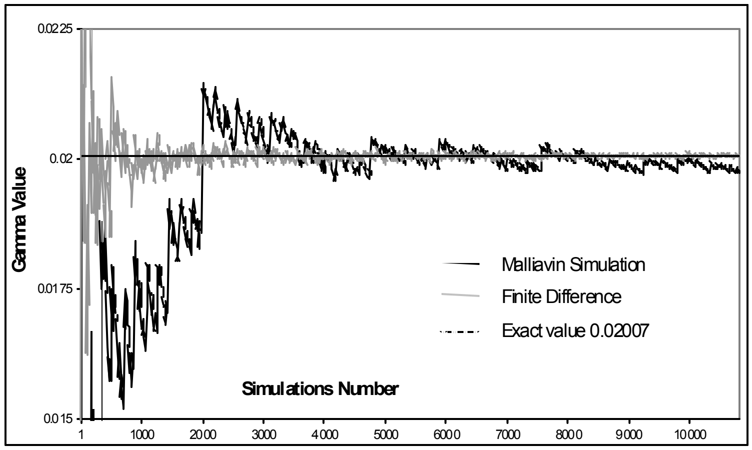

Figure 6: Comparison of the computation of the Gamma of a Binary option by finite differences and by Malliavin weighted scheme

4.2 Typology of options requiring Malliavin weighted scheme

As shown by simulation examples, a paradox of the Malliavin weighted scheme is the implication of the payoff function on the method. The weighting function does not rely on the payoff function. However, its comparative advantage does depend on the form of the payoff. Indeed, the finite difference method are crucially related to the form of the option payoff. A natural and very interesting open problem is to classify the option types for which the Malliavin weighted scheme should be preferred. In this section, we precisely try to define a typology of options for which the Malliavin technology outperforms the traditional finite difference method. We can make many remarks:

- the Malliavin weighting function is independent from the option payoff. This indicates that the disturbance caused by the weighting function is not influenced by the payoff. This is not the case of the finite difference method for which the payoff function matters crucially.
- the weighting function explodes for small maturities. This suggests that the Malliavin technology is inappropriate for small maturities options.
- the computation of the gamma is similar to the vega since in the case of the Black and Black Scholes model, there is a direct proportionality between the gamma and vega coefficient. The proportionality can be read on the weighting function, whereas it is not obvious in the finite difference method. A standard finite difference method would lead to compute the gamma by the finite difference approximation
$$\Gamma \approx \frac{\text{Price}\left(S_0 + dS_0, \sigma\right) - 2\text{Price}\left(S_0, \sigma\right) + \text{Price}\left(S_0 - dS_0, \sigma\right)}{dS_0^2}$$
as well as the vega $v \approx \frac{\text{Price}(S_0,\sigma+d\sigma)-\text{Price}(S_0-,\sigma-d\sigma)}{2d\sigma}$.
- the Malliavin technology in the case of the gamma reduces a second order differentiation to no differentiation. This implies that the efficiency of the Malliavin method is enhanced in the case of the gamma compared to the delta or even the rho.

Before, giving an empirical typology of option, we quantified the variance reduction induced by the Malliavin method. And we can claim that Malliavin based formula is a variance reduction technique. This is well illustrated by table 1, where we have given the ratio between the estimated volatility of the Malliavin simulation and the finite difference simulation. We can see that the method is more efficient for gamma, than for vega.

The variance reduction is of comparable order for delta and rho. The number of simulation draws for the table 1 was 20,000 as indicated by N=20,000. We give the ratio of simulation variances between finite difference and Malliavin-based simulation. Since the variance decreases roughly linearly in n, a ratio of ten means that we need to do 10 n draws in the finite difference method to get the same variance as the one obtained by the Malliavin method with only n draws.

We found that for the corridor option, the Malliavin weighted scheme, when compared to finite difference, improved the computation of the Greeks by a factor bigger than 100 for the case of the delta, 6000 for the case of the gamma, 33 for the case of the rho and 5900 for the case of the vega as stated by table 1. These are big numbers. It means for example that we need about 6 millions of draws to compute the gamma with the finite difference method to get the same accuracy as a simulation based on a Malliavin weighted scheme.

The faster convergence of Malliavin weighted scheme over the finite difference method with common random numbers comes from the fact that the Malliavin method avoid differentiating and smoothens considerably the payoff of the option to simulate.

Option type	Variance ratio $\widehat{\sigma}^2_{\text{Finite Difference}}/\widehat{\sigma}^2_{\text{Malliavin}}$	delta	gamma	rho	vega
Call	N=20,000	0.1273	0.1272	0.401	0.0735
Binary	N=20,000	7.15	4916	6.56	81
Corridor	N=20,000	144.98	6864	33	5920

Table 1: Comparison of the Malliavin weighted scheme and the finite difference method

Summarizing all the results given by the simulations, we draw the following conclusions:

- The Malliavin method is appropriate for option for which the mean-square convergence of a shifted option $P(X_0 + \varepsilon)$ to the normal one $P(X_0)$ is linear in ε. This is the case of any option with a payoff expressed as a probability that a certain event occurs conditionally to the underlying level at a certain time. This is the case of any binary and corridor option.

- The maturity of the option is a crucial factor for the Malliavin method since it leads to an exploding weighting function. However, the traditional method underperforms as well.

- The Malliavin method leads to weighting functions which are roughly (polynomial) functions of the Brownian motion. The variance of the weighting function increases for high values of the Brownian motion. To get the Greek, we multiply the weighting function by the option payoff. This implies that if the payoff function is very small for high value of the Brownian motion, the variance is going to be low. This indicates that Malliavin formulae are more efficient for put than call options. Two remarks should be made. First, it is more appropriate to use the put-call parity and therefore to calculate Greeks only for a put. Second, we can use a mixed strategy referred as the Malliavin formula and explained in the next subsection.

4.3 Local Malliavin formula

The intuition behind the integration by parts is to smoothen the payoff at the discontinuity kink. However, there is no advantage in using the Malliavin formula when the payoff is smooth. This hints at using a mixed strategy. At the discontinuity, we use an integration by parts by means of the Malliavin formula. Otherwise, we use the traditional finite difference. The finite difference method contains in a way a natural antithetic variable variance reduction when it is implemented with common numbers as explained in section 2.2, page 4. Let us describe the idea on the case of the delta of a call. We have seen that the delta can be written as the expected value of a payoff times a weighting function (section 3.1, page 10), which in the simple case of the Black Scholes framework leads for a European option to

$$\delta = \mathbb{E}\left[e^{-\int_0^T r_s ds}(X_T - K)^+ \frac{W_T}{x\sigma T}\right]$$

The weighting function is multiplied by the term $(X_T - K)^+$ which is big for large values of X_T, corresponding to large values of the Brownian motion W_T. This generates some increased variance because of the weighting function $W_T/x\sigma T$. When X_T is "large", W_T is "large" and therefore $(X_T - K)^+ * W_T/x\sigma T$ is even "larger" with a substantial variance. Writing the delta as the sum of two terms

$$delta = \frac{\partial}{\partial x}\mathbb{E}\left[e^{-\int_0^T r_s ds}(X_T - K)^+ 1_{\{K<X_T<K+\varepsilon\}}\right] + \frac{\partial}{\partial x}\mathbb{E}\left[e^{-\int_0^T r_s ds}(X_T - K)^+ 1_{\{K+\varepsilon\leq X_T\}}\right]$$

Using the Malliavin integration by parts only for the first term, and interchanging the expectation and the differentiation operator for the second, we come to

$$delta = \mathbb{E}\left[e^{-\int_0^T r_s ds}(X_T - K)^+ 1_{\{K<X_T<K+\delta\}}\frac{W_T}{x\sigma T}\right] + \mathbb{E}\left[e^{-\int_0^T r_s ds}1_{\{K+\delta<X_T\}}Y_T\right]$$

ndeed, it is very efficient to take a small localization parameter like $\varepsilon = 1$. In this case, it leads to a reduction of variance, that is to say $\widehat{\sigma}^2_{\text{Malliavin}}/\widehat{\sigma}^2_{\text{LocalMalliavin}}$ =13.88 and a variance reduction from finite difference to local Malliavin of 1.77, that is to say $\widehat{\sigma}^2_{\text{Finite Difference}}/\widehat{\sigma}^2_{\text{LocalMalliavin}}$ =1.77. Therefore, the Malliavin local formula is more efficient than the standard finite difference method. The factor of 1.77 means intuitively that we need a simulation of 17,000 draws with a finite difference approximation to get the accuracy of a 10,000 draws simulation with the local Malliavin based formula. Indeed, the crucial point in this formula is to find an interesting value of ε, taken here as 1% of the underlying initial level. It would be nice to examine the impact of this parameter on the variance reduction with some theoretical considerations. That is one of the promising future area of research for the Malliavin technique.

5 Conclusion

In this paper we have seen that using Malliavin calculus and its integration by parts formula, we can smoothen the function to be estimated by the Monte Carlo or Quasi Monte Carlo procedure. This outperforms traditional finite difference method in the case of digital option as well as corridor, with a gain on the variance of the simulation of more than 4900 and 6800 for the gamma of respectively the digital and the corridor option. These are big numbers since this means that we roughly need to do a finite difference simulation of 4,900,000 respectively 6,800,000 draws to get the same accuracy as a Malliavin simulation based on 1,000 draws.

However, we recommend a cautious use of the Malliavin formula. It turns out to be very efficient for discontinuous functions like a digital, corridor payoff function. However, for smooth functions, it can handle the computation of the Greeks more inefficiently than a finite difference method. This is because the finite difference method includes an antithetic variate variance reduction method. We suggest to use a local version of the Malliavin method, so as to smoothen the payoff at the kink and elsewhere to use finite difference method with common random numbers. Other relationships like put-call parity should be used as well.

There are many extensions to this paper, especially to more complicated models than the Black one. An interesting enlargement is the advanced study of the local Malliavin method. As a conclusion, we conjecture that the Malliavin method is going to have an increasing influence over the next years since it is a powerful method to compute the Greeks. Last but not least, we should also mention that there exists extensions to conditional expectation and American options as explained in Fournié et al. and Lions, Régnier (2000).

A Vega and Rho for the Corridor option

We decided to study the delta and gamma to compare different type of options. However, the efficiency (or not) of the Malliavin weighted scheme for the rho and vega is similar to the case of the delta and gamma.

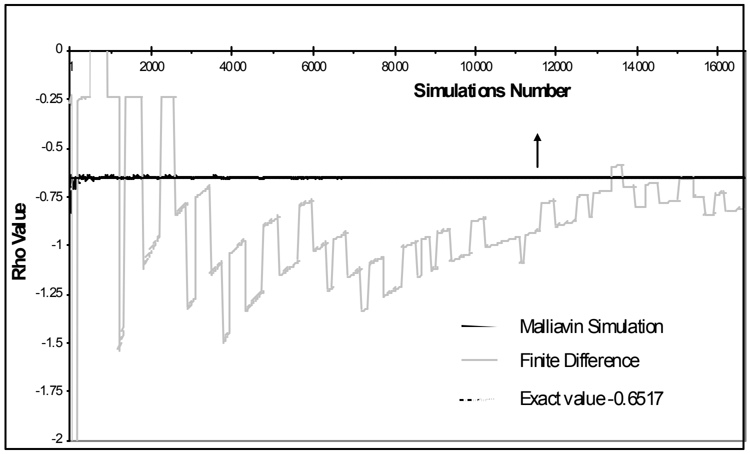

Figure 7: Efficiency of the Malliavin weighted for the computation of the rho of a Corridor option. The pseudo periodicity of the finite difference comes from the pseudo periodicity with which the Sobol sequence fills the space

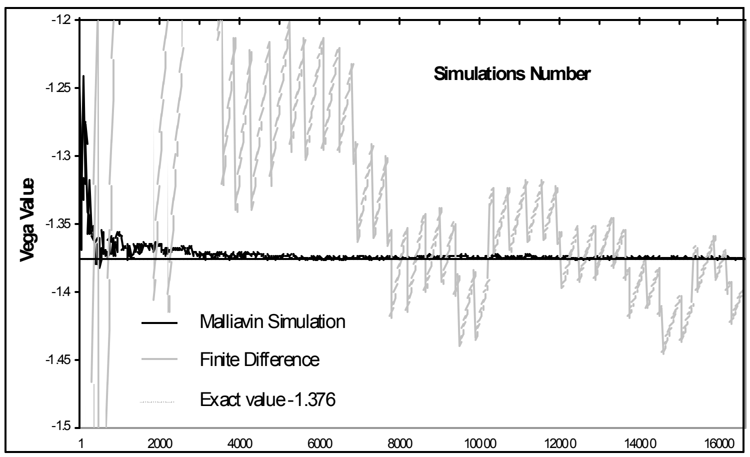

Figure 8: Efficiency of the Malliavin weighted for the computation of the rho of a Corridor option. The case of the vega is very similar to the one of gamma since these two sensitivities are proportional in the case of a single option in a Black Scholes framework

B Primer on Malliavin calculus

The objective of this short primer is to give an intuitive presentation of Malliavin calculus. For a more rigorous and detailed explanation, we refer the reader to the excellent book of Nualart (1995). Malliavin calculus is a synonym of calculus of variation of stochastic processes. Even if its original motivation was to provide a probabilistic proof of the existence and smoothness of solutions of particular PDEs (the of Hormander's sum of squares theorem), Malliavin calculus has turned out to be a very powerful tool for giving other representation of stochastic processes, allowing to prove certain properties of stochastic processes (especially smoothness conditions). Because the Brownian motion is not differentiable in the traditional sense, Malliavin calculus defines a derivative, using a local perturbation on the Brownian motion and more generally on a martingale process. the Malliavin derivatives is in a sense the Gateau derivatives along a bumped path. It measures in a sense the impact of bumping locally the Brownian path. Let us take a function of the Brownian motion , $(W_t)_{t\geq 0}$: $F : t \rightarrow F(W_t)$. Let us bump the Brownian motion only locally at a time s. In mathematical terms, the perturbed Brownian motion is the superposition of the original Brownian motion and a bump at time s of size ε: $W_t + \varepsilon 1_{\{s\leq u\}}$. The Malliavin derivative is defined intuitively as

$$D_s F : t \rightarrow \lim_{\varepsilon\rightarrow 0} \frac{F\left(W_t + \varepsilon 1_{\{s\leq u\}}\right) - F(W_t)}{\varepsilon} \tag{B.1}$$

where the limit can usually be interpreted as a.s. This trivially leads to the Malliavin derivative of a Brownian motion given by the indicative function: $D_s W_t = 1_{\{s\leq u\}}$

The interest of the Malliavin calculus is to satisfy usual derivation rules:

- Chain rule for compound function, $\Phi : t \rightarrow G(F_1(W_t), F_2(W_t))$

$$D_s\Phi = \sum_{i=1} \frac{\partial}{\partial x_i} G.D_s F_i \tag{B.2}$$

- Integration by parts, (or duality between the Malliavin derivative and the Skorohod integral).

$$\mathbb{E}\left[\int D_s F u ds\right] = \mathbb{E}\left[F\partial(u)\right] \tag{B.3}$$

  where $\partial(u)$ is called the Skorohod integral. This relation is the cornerstone formula as it enables to smoothen the function inside the expectation. Intuitively, the Skorohod integral could be compared to the divergence operator[8] (up to the minus sign) as for deterministic function on $(\mathbb{R}^n, \lambda^n)$, we have

$$\int_{R^n} \langle \nabla f, u\rangle_{R^n} d\lambda^n = -\int_{R^n} divud\lambda^n$$

- Skorohod integration: for adapted processes u, the Skorohod integral coincides with the Ito integral

$$\delta(u) = \int u dW_s \tag{B.4}$$

  Moreover, the Skorohod integral satisfies some properties for the product of two functions:

$$\delta(Fu) = F\delta(u) - \int D_t F u dt \tag{B.5}$$

- Malliavin derivatives of a jump-diffusion: Let $(X_t)_{t\geq 0}$ be defined by its jump-diffusion equation:

$$X : dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t + \lambda(t)\, dJ_t$$

  with initial condition $X_0 = x$. And let us define its first variation process (also called the tangential process) $(Y_t)_{t\geq 0}$ defined as , $Y_t = \frac{\partial}{\partial x} X_t$,. obviously $Y_0 = 1$ and

$$Y : dY_t = \frac{\partial}{\partial x} b(t, X_t)\, Y_t dt + \frac{\partial}{\partial x}\sigma(t, X_t)\, Y_t dW_t$$

[8]Some authors refer to the Skorohod integral as the stochastic divergence operator.

the Malliavin Derivatives of $(X_t)_{t\geq 0}$ is then given by

$$D_s X_t = \sigma(s, X_s) Y_t Y_s^{-1} 1_{\{s\leq t\}} \tag{B.6}$$

Let us conclude by saying in passing that the Malliavin derivative satisfies standard rule of derivation, namely for a product, we have

$$D_t(FG) = D_t F.G + F.D_t G \tag{B.7}$$

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