Delta equivalent asset flows

The delta equivalent asset flows refers to the quantity to buy in a given asset to have an overall neutral delta position. Current market practice is to delta hedge with futures, forward or spot assets. However, one may also want to use two options to get a delta neutral position. In that case the delta equivalent asset flow between option 1 and option 2 would be the quantity of option 2 to have the same delta as 1 option 1. More generally, one calls delta equivalent asset flows for a given portfolio and a given asset the quantity of asset that has the same delta as the portfolio. Because already determining accurately[1] a delta is a complex problem, we shall first review the concept of efficient delta before looking at the implication of delta equivalent asset flows hedging and other risk management usage like Value at Risk, Delta VAR and Stress testing.

What is a good delta?

Getting the right delta for a given option is not an easy problem. Defined as the sensitivity of option prices with respect to the change in the underlying prices, the futures prices or the forward one, the delta is one of the key concepts in option trading. In Black Scholes, delta can be easily computed by taking the first order derivatives of the option price with respect to the spot, leading for vanilla options to the famous

$$e^{-qT} N(d_1) \qquad (1.1),$$

with

$$d_1 = \frac{\ln(S_0 / K) + (r - g - \sigma^2 / 2)T}{\sigma\sqrt{T}} + \sigma\sqrt{T} \qquad (1.2),$$

where $N(x)$ is the cumulative normal density function, S_0 is the spot stock price, K the strike price, r the risk free rate, q the continuous yield dividend, T the option maturity and σ the Black Scholes implied volatility.

But life is not as simple as Black Scholes and assuming log normal dynamics for the asset price can be seriously misleading for the delta.

In fact, if one assumes that the volatility $\sigma(S)$ depends on the spot as in local volatility models, we get that the right delta should be equal to the Balck Scholes delta $\frac{\partial}{\partial S}C(S,\sigma(S))$ plus the vega $\frac{\partial}{\partial \sigma}C(S,\sigma(S))$ times the slope of the smile $\frac{\partial}{\partial S}\sigma(S)$. Mathematically, this results comes from compound derivation:

$$\frac{\partial}{\partial S}C(S,\sigma(S))=\frac{\partial}{\partial S}C(S,\sigma(S))+\frac{\partial}{\partial \sigma}C(S,\sigma(S))\frac{\partial}{\partial S}\sigma(S) \tag{1.3}$$

Sofar, we have only relaxed the assumption that the diffusion is lognormal and use a local volatility model to account for better reality. But what if stocks do not trade continuously. Can we still provide an efficient delta? We should be able to use a Merton model (1976) to account for the jump of the price. This models assumes that the price dynamics is given by a lognormal diffusion plus a jump:

$$\frac{dS_t}{S_t}=\{r_t-\lambda_t(EJ-1)\}dt+\sigma dW_t+(J_t-1)dN_t \tag{1.2}$$

[1] This means that one needs to take into account the imperfections of Black Scholes pricing model.

where jumps occurs according to a Poisson process N_t with time dependent intensity λ_t, with jump size $J_t - 1$, where the variables J_t are identically and independently distributed with a lognormal density with its log with a mean m and variance v^2. Consequently, $E[J] = \exp\left(m + \frac{1}{2}v^2\right)$. This leads to a delta of the form:

$$\Delta = \exp\left(-\int_0^T \lambda_s E(J)ds\right)\sum_{n=0}^{\infty}\frac{\left(\int_0^T \lambda_s E(J)ds\right)^n}{n!} S_0 \exp(-gT)N\left(d_1^{\ n}\right) \qquad (1.3),$$

with
$$d_1^{\ n} = \frac{\ln(S_0 / K) + \int_0^T \left(\mu_n(s) + \frac{1}{2}\sigma_n^{\ 2}(s)\right)ds}{\sigma_n(s)\sqrt{T}} \qquad (1.4),$$

with
$$\mu_n(s) = r_s - \lambda_s(E(J) - 1) + n\frac{\left(m + v^2 / 2\right)}{T} \qquad (1.5),$$

and
$$\sigma_n(s)^2 = \sigma(s)^2 + n\frac{v^2}{T} \qquad (1.6).$$

Sofar so good, but what if we want to account for transaction costs? Standard theory assumes that the option trader can trade at no cost unlimited quantity of the stocks. Clearly for barrier options or for options on illiquid products like fund options, this is far from beyond the case. Trading according to Black Scholes would ruin the option trader. Again, we could find another model, in fact the Leland (1985) model, to account for the transaction costs. In the Leland model, the portfolio is revised every δt where δt is a finite and fixed time step. Transaction costs are proportional to the value of the transaction of the underlying, hence a transaction cost of the type $\kappa|v|S_t$ where v is the number of shares bought or sold by the investor. His result is to show that long option position on call or put should be valued with an adjusted volatility of the type

$$\sigma_{low} = \sigma\left(1 - 2\sqrt{\frac{2}{\pi\delta t}}\frac{\kappa}{\sigma}\right) \quad (1.7),$$

while short position on calls or put valued with an adjusted volatility of the type

$$\sigma_{up} = \sigma\left(1 + 2\sqrt{\frac{2}{\pi\delta t}}\frac{\kappa}{\sigma}\right) \quad (1.8).$$

But since the transaction costs in the Leland (1985) model are not very realistic, how would we help our dynamic hedger? Best practice is to take a simple model and know what you are doing and hence taking some reserve for additional risk not accounted in the model.

For instance, if we are pricing cliquet type structure, how can we make sure that we have a consistent term structure of forward volatility for our delta? Similarly, if we are pricing a hybrid structure, depending very much on the cross asset correlation, how can we make sure that the price but also delta is at the market price and that we did not forget or miss some extra risk? Forward smile and correlation share the same problem of being not tradable risk, hence very model dependent. The theoretical argument of using parabolla or log normal contracts to lock-in forward volatility can only give us an vague estimation of the forward volatility. However, this would not provide any forward volatility information for a given strike. Similarly, correlation can not be locked in. Extra flexibility in term of correlation, like dynamic copulas (term structure of copulas) can help to capture a model with smooth delta.

So to cut it short, after finetuning the model, we would still be left with some model risk assumptions. We cannot guarantee that the model is for sure realistic. And as we have seen, the dynamic of the price makes quite a subtantial difference on the delta. In brief, getting a good estimate of the delta is very much an art and traders should be careful about the potential bias introduced by a given model.

Mean reverting and trending markets

If we know that markets are mean reverting, can we use this fact to get a cheaper delta? Yes, we can but at the risk of making a bet on the mean reversion. If markets are mean reverting, a smart trader may calculate try to calculate a delta hedge that is only re-balancing the portfolio in timely way. First of all, the trader should quantify the transaction costs. Hedging very frequently would kill the profit of the trader, as the trader would hedge back and forth, and loose a substantial amount of money in transaction costs. Clearly this is not optimal. In this case, the appropriate delta would be the one that provide the risk minimizing strategy with the lowest cost, knowing the transaction cost and a given mean reversion. Like for the capital asset pricing model, there would be a trade-off between re-balancing frequently to perfectly hedge and a trading strategy that takes into account the mean reversion and therefore hedge less frequently but can bear some risk due to the stochasticity around the mean reversion. With simple assumption, the problem can be expressed as an optimization problem, where the solution has to solve a specific Hamilton Jacobi Bellman equation.

On the contrary, if markets are trending, the trader should be better off using a delta that incorporates in a sense some gamma. She knows there is a high chance that a normal delta hedge would need to be rebalanced when the asset has moved in direction of the trend. The trader would therefore need to calculate a shadow delta that accounts for this expected rebalancing of the hedge.

Entry category: options

Scope: risk management, model risk, using mean reverting and trending markets.

Related articles: asset-flow mapping, delta.

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