Wiener process

Wiener process is the other name for a Brownian motion process. It designates a stochastic process whose increments are independent, stationary and normally distributed. There are many books about wiener processes. In this article, we aim at providing the underlying intuition of a Wiener process as well as the useful tool to know to use Wiener processes.

SOME HISTORY

The Brownian motion was *first observed in 1828* and 1829 by the Scottish *botanist Robert Brown*, when looking at the trajectory of pollen particles in suspension under a microscope. He observed that the trajectory was in constant irregular motion. The Brownian motion was first introduced in finance in the thesis of the *French Louis Bachelier in 1900*. Bachelier considered Brownian motion as a possible model for stock market prices, early before any mathematical option pricing theory has been developed. However, his work did not take up. It is only *in 1905, when Albert Einstein* produced some work in the area of diffusion transport, that the study of Brownian motion was identified as an important research topic. Using Brownian motion as a model of particles in suspension, he estimated the Avogadro's number 6.10^23 based on the diffusion coefficient D in the Einstein relation Often in science, the Brownian motion was invented before having some rigorous foundation. It was only in 1923 that the existence and construction of Brownian motion was established by *Norbert Wiener*. Wiener process becomes a synonym of Brownian motion and the measure was called the Wiener measure in his honour. And it is *Samuelson in 1969* that resurrected Brownian motion in finance. Later, *the seminal work of Black and Scholes (1973)* established Brownian motion as a standard assumption for continuous time modelling.

INTUITION ABOUT BROWNIAN MOTION

Any variable whose values are uncertain over time is said to be stochastic or random. Firstly, it is classified as discrete-time or continuous-time process according to the way it changes with respect to time. Discrete-time process changes only at certain fix points in time whereas continuous time process changes at any time. Secondly, a continuous time stochastic process is classified as a discrete or continuous variable according to its change. Discrete variable, jump or point processes change discretely (they have discontinuous jumps in their trajectory), whereas continuous or diffusion processes can take any value within a range. A mixture of the two types is called a jump-diffusion process and is a very general type of process.

Brownian motions are *continuous time stochastic process evolving continuously*.

A process is said to be Markovian if its futures does only depend on its present status and not the past. A process is said to have independent increments if its increments are not depending on previous past increments. Increments of a process are stationary if their evolution is ruled by the same laws, or more precisely they have the same distribution.

A minimalist definition of a Brownian motion is a continuous-time process, with continuous values, with independent and stationary increments. Because of the independence of its increments, it is obviously a Markov process. The interesting characteristics of Brownian motion are its generality and its tractability.

A process is said to be a Martingale if the expected value of its future realization is equal to its present value. The Brownian motion is also a martingale.

The interest of the Wiener process lies in its generality. One can almost say that at first order, any stochastic process behaves pretty much like a Brownian motion. In fact, Wiener processes are the asymptotic limit of many random variables (and more generally diffusion processes). This fact results is a consequence of the Central limit theorem or Moivre-Laplace theorem (Abraham de Moivre 1667-1754 and Le Marquis de Laplace 1749-1827), which states that the sum of independent and identically distributed variables converges under certain very general conditions (existence of the first two moments) to a normal distribution or Gaussian distribution. And precisely, the Brownian motion is a normal process. As a consequence, the result of a random walk converges to a Brownian motion. This explains why random walks are sometimes taken as a synonym of Brownian motion.

Extrapolating from the theory, this means that the trajectory of a drunk ward trader in Manhattan has unconsciously a similar behaviour to stock asset evolution. Because of this asymptotic limit, it is often quite appropriate to model the evolution of a variable, after some transformation, by a Brownian motion.

Second, the Brownian motion has explicit formulae for the density of its running extremum, minimum and maximum, its law conditional to having not breached a certain threshold. Its cumulative density function is easily computed numerically, while its trajectory is easy to simulate (Monte Carlo simulations). In option pricing theory, it is therefore possible to get closed forms for many exotic products.

TECHNICAL TOOLS ABOUT BROWNIAN MOTION

Let us denote by $(W_t)_{t\in R}$ a Brownian motion. We have the following property

- For $0 < t_1 < ... < t_n$, for $i > 1$ and $j > 1$ with $i \neq j$ $\left(W_{t_i} - W_{t_{i-1}}\right)$ and $\left(W_{t_j} - W_{t_{j-1}}\right)$ are two independent variable distributed according to a normal distribution with zero mean and a variance equal to $t_i - t_{i-1}$ respectively $t_j - t_{j-1}$
- The probability of a given realization is easy to compute:

$$P\{W_t \in [x, x+dx]\} = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx \,,\ P\{W_t < \mathrm{a}\} = N\left(\frac{a}{\sqrt{t}}\right)$$

where $N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left(-x^2\right) dx$ is the cumulative density function

- the covariance between two realization of the Brownian motion is simply given by $E[W_t W_s] = Min(t, s)$

- the n moment of a Brownian motion is easy to compute and given by $E\left[W_t^{2n}\right]=\frac{(2n)!}{2^n n!}t^n$ while obviously $E\left[W_t^{2n+1}\right]=0$
- the asymptotic behaviour of the Brownian motion is well known. In particular, we have
- $\lim_{t\to\infty}\frac{W_t}{t}=0$, almost surely
- $P\left[\sup_{t>0} W_t=+\infty, \inf_{t>0} W_t=-\infty\right]=1$
- the Brownian is almost surely nowhere differentiable.
- the Markov property implies in particular the reflexion principle:
- Let a be a real number represented some threshold, let $(W_t)_{t>0}$ be a Wiener process, then the reflected process defined as W_t while the Brownian motion has not reached the threshold and $2a-W_t$ as soon as it has reached the threshold is still a Brownian motion.

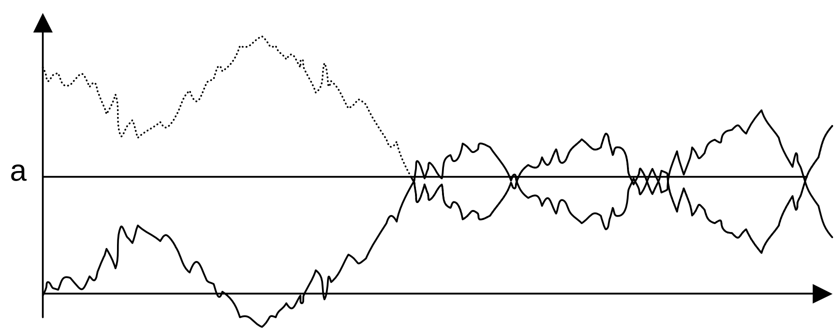

Figure 1: Reflected Brownian motion

The reflection principle is the key tool to compute the law of the running maximum (and minimum) of a Brownian motion. In particular, one can then easily show that the joint law of the Brownian motion W_t and its running maximum M_t is given by

$$P\left(W_t \in [x, x+dx], M_t > a\right) = \begin{cases} 1/\sqrt{2\pi t}\ \exp\left(-(2a-x)^2/2t\right)dx & \text{for } x < a \\ 1/\sqrt{2\pi t}\ \exp\left(-x^2/2t\right)dx & \text{for } x > a \end{cases}$$

hence the following law for the running maximum:

$$P\left(W_t \in [x, x+dx], M_t \in [y, y+dy]\right) = \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y-x)^2}{2t}\right) \qquad \text{for } 0 \le x \le y$$

Entry category: Mathematical models

Related articles: Stochastic processes in financial markets; Ito's lemma

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[1] The views and opinions expressed herein are the ones of the author's and do not necessarily reflect those of Goldman Sachs

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