

Tanaka formula and Levy process

Simply speaking, the Tanaka formula is an extension of the Itô formula while Lévy process is an extension of Brownian motion. Because the Tanaka formula and Lévy process are two different concepts, we will explain them separately. Since the goal of this article is to introduce the concepts, we will present them in a very accessible way referring to reference books like Karatzas and Shreve (1998) or Revuz and Yor (1994) for Tanaka formula and Sato (1998) for Lévy processes.

1) Tanaka formula

The standard theory of stochastic calculus is mainly based on the Itô lemma. This lemma states that the differential function of a twice differentiable function of a stochastic process described by a diffusion equation is given by the normal terms in the deterministic case plus an extra terms equal to the second derivatives function times the quadratic variation of the process. Let $(W_t)_{t \in \mathbb{R}}$ be a standard Brownian motion (or Wiener process, see Wiener processes) and $(X_t)_{t \in \mathbb{R}}$ a diffusion process whose evolution is described by a diffusion equation, whose coefficients $\alpha(t, X_t), \beta(t, X_t)$ are Lipschitz function of both t and X_t :

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t \quad (1.1).$$

The derivative function of a function $f(t, X_t) \in C^2$, twice differentiable with second order derivatives function continuous, is given by:

$$df(t, X_t) = \frac{\partial}{\partial t} f(t, X_t) dt + \frac{\partial}{\partial x} f(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(t, X_t) \beta^2(t, X_t) dt \quad (1.2)$$

However, Itô formula does not apply when the function $f(t, x)$ is not C^2 . And most of the option payoffs are not C^2 . Even simple options' payoffs like the ones of calls or put do not fulfil the condition of the Itô lemma. However, one can show that the Itô formula can be extended to function that are only convex. The formula first proved by Hiroshi Tanaka and completed by Watanabe, uses the concept of local time and distribution theory to do the generalisation. The second order term of the Itô formula is replaced by the formal second order derivatives (in the sense of distribution) times the occupation density often called in probability theory the local time. If $f(t, x)$ is a convex function, its differential function is given by:

$$df(t, X_t) = \frac{\partial}{\partial t} f(t, X_t) dt + \frac{\partial}{\partial x} f_-(t, X_t) dX_t + \frac{1}{2} \left(\int_{-\infty}^{+\infty} \frac{\partial^2}{\partial x^2} f(t, x) \frac{\partial}{\partial t} L_t(x) dx \right) dt \quad (1.3),$$

where $\frac{\partial}{\partial x} f_-(t, X_t)$ is the left limit of the first order derivatives of f with respect to x , L_t^x is the local time of the process X_t in x defined as the limit of the occupation time in x :

$$L_t^x = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} P(|X_s - x| \leq \varepsilon, \text{ for } 0 \leq s \leq t) \quad (1.4)$$

First, it is worth noting that as for the Itô formula, this can be extended easily to jump diffusion: if X_t is a jump diffusion driven by a Poisson process N_t with a jump size of J_t (see Poisson process and jump diffusion),

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t + J_t dN_t \quad (1.5)$$

The additional terms comes from the jump part leading to:

$$df(t, X_t) = \frac{\partial}{\partial t} f(t, X_t)dt + \frac{\partial}{\partial x} f(t, X_t)dX_t + \frac{1}{2} \left(\int_{-\infty}^{+\infty} \frac{\partial^2}{\partial x^2} f(t, x) \frac{\partial}{\partial t} L_t(x) dx \right) dt + \int_J (f(t, X_t + J_t) - f(t, X_t)) dN_t \quad (1.6)$$

The concept of local time has a very interesting financial interpretation: it is the time value of an option. Indeed, in the case of a diffusion of the type Dupire, applying the Tanaka formula to the payoff function of a call of maturity T and strike K and taking the risk neutral expectation leads to:

$$C(T, K) = B(0, T)(S_0 - K)^+ + B(0, T)L_T(K) \quad (1.7),$$

which states that a call (similarly a put) option is equal to its intrinsic value plus the discounted local time in the strike of the process. The discounted local time is therefore equal to the time value of the call (similarly put) option.

The Tanaka formula is useful to prove the Dupire model. It has also been successfully used to show how to find a static hedge for barrier options when assuming a local volatility model (see Carr or Andersen et al.), how to compute passport option and relate it to lookback options (Henderson and Hobson).

2) Lévy process

Roughly speaking, Lévy processes are an extension of the Brownian motion. Lévy processes are general stochastic processes, with stationary independent increments. Examples are Brownian motion, Poisson processes, compounded Poisson process, stable processes (such as Cauchy processes), and subordinators (such as Gamma-processes). They form a basic class in stochastic analysis.

Processes that are derived from a Lévy process encompass all the diffusion process derived from a Brownian motion. Lévy processes can be characterised by their Lévy-Khintchine representation of infinitely divisible distributions. The Lévy-Khintchine formula says that a Lévy can always be represented as the convolution of a random Gaussian variable (possibly with a rather warped covariance matrix), with a compound Poisson random variable (possible with infinite intensity measure). In short, if X_t is a Lévy process, its characteristic function can be written in the form of a Brownian part and a Poisson term:

$$E[\exp(i\lambda X_t)] = \exp\left(i\lambda\mu - \frac{1}{2}\lambda^2\sigma^2 - \int_R (1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}})\Pi(dx)\right) \quad (2.1),$$

where Π is a positive measure on R so that $\int_R \text{Min}(1+x^2)\Pi(dx) < +\infty$, referred to as the Lévy measure.

Academic research have recently focused on Lévy processes (as opposed to Brownian motion). Indeed, Lévy processes are more general processes than standard Brownian motion and already encompasses lots of well know processes (like the Variance gamma process of Madan and Seneta (1990)

and many of the simple jump diffusion models (like the Merton model). Lévy processes are a way to incorporate the non log-normality of option underlying (referred to as the volatility smile as the function of the Black Scholes implied volatility shows some smiley figures) within the random process itself. This stands in contrast to stochastic and deterministic volatility models (See Heston (1993), Dupire (1994), Derman (1994) etc).

The pricing of options when assuming Levy processes can be done either by solving the partial integro differential equation (Eberlein (1995) (very similarly to the solving of a jump diffusion models) or by using Lévy exponent and the Laplace transform of the option price. (Benhamou (2001)).

Models driven by Lévy processes have been derived both for equity derivatives but also for interest rates derivatives, generalising the Heath Jarrow Morton condition to this wider class of stochastic processes.

Interestingly, a process driven by a Lévy process can be shown to be a particular representation of any (non-singular) jump diffusion process. However the use of the property of Lévy processes makes both theoretical and numerical solving easier. Last but not least, one can derive closed forms for barrier options for process written as the exponent of a Lévy process

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Entry category: mathematical models

Scope: options

Related articles: Brownian motion; Ito's lemma, Jump diffusion.

¹ The views and opinions expressed herein are the ones of the author's and do not necessarily reflect those of Goldman Sachs

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