In-arrears swap

Also known as a delayed reset swap, an in-arrear swap is an interest rate fixed for floating swap that has its floating leg that pays at the regular payment date a rate that has just reset (usually that has reset two business day ago for Euro JPY and USD swap and that has just reset for GBP swap). In the case of swap paying every six months, the reset rate at the payment date would be fixed six months and two days ago in a regular swap only two days ago in the in-arrear version. In an in-arrear, since the reset rate paid is not paid after its reference period as in a standard swap, the floating leg cannot be valued as the sum of the forward Libors but has to take into account the volatility of the forward rates via an adjustment called the convexity correction. We will give more details when examining the pricing of in-arrear swaps.

MARKETING OF IN-ARREAR PRODUCTS

In-arrear swaps are popular products in a steep yield curve environment to a fix rate receiver who thinks that short term rates will not rise as fast as the yield curve predicts, pocketing up the difference between the fix rate of the standard swap and the one of the in-arrear swap known as the pick up, while still paying low Libor resets.

Usually, clients (corporates or financial institutions) receive fix and pay floating. In a steep yield curve environment, because of the delayed resets, an in-arrear swap has a fixed coupon much higher than the corresponding swap, making it attractive. It can be as high as 50 basis points in certain situation. If the investor/trader thinks that rates will not rise as fast as the yield curve predicts, in an-in arrear swap, he will monetize the difference between the expected rise of the short term rates (reflected by the high fixed rate) and the real movements of these rates.

PRICING

The pricing can be done either by using standard arguments of forward neutral pricing measure or by using static replication with a set of caplets. Let us review the two methods that lead to similar result although the second one has the strong advantage to show the static replication portfolio and to be model free.

In a regular fixed for floating swaps, the reset dates are called to be in advance while the payment dates are said to be in-arrear. To be accurate about date conventions (and quants know that the devil is in this small details), a swap contract has reset dates and payment dates (see master agreements). Reset dates are dates used to fix the rate used for the computation of the floating amount, period end dates are used to compute the accrual while payment dates are the days when the floating amount is paid.

In a standard swap, paying fixed yearly m times, paying floating every six months with $n = 2m$ payments, resetting two business days in advance[1] and with an effective date T_0, reset dates are T_0, $T_1 = T_0 + 6m$, …, $T_{n-1} = T_0 + (n-1)*6m$, period end dates are T_1, T_2, …, $T_n = T_0 + n*6m$, while

[1] These are the default conventions for Euro swaps.

payment dates are $T_1 + 2bd$, $T_2 + 2bd$, …, $T_n + 2bd$, where *2bd* stands for two business days (according to the calendar of the floating leg). In the corresponding in-arrear swap, reset dates are shifted by one index to become T_1, T_2, …, T_n. Similarly, period end dates are also shifted by one index while payment dates would remain unchanged. Let U_1, U_2,…, U_m be the fixed leg payment dates, v_i the corresponding accrual terms and K the fixed rate. Denoting by $D(0,T)$ the discount factor from 0 to time T, $F_{i+1}(T)$ the value at time T of the Libor that resets at time T_i and whose interest period is from T_i to T_{i+1}, τ_{i+1} its accrual, the payoff of the in-arrear swap paying fixed receiving floating, can be expressed as the sum of its floating leg minus its fixed leg

$$E\left[\sum_{i=1}^{n} D(0,T_i + 2bd)\tau_{i+1}F_{i+1}(T_i)\right] - E\left[\sum_{i=1}^{m} D(0,T_i + 2bd)v_i K\right] \quad (1.1)$$

where E represents the risk neutral expectation. The valuation of the fixed leg is straightforward (exactly the same as in a regular swap and equal to $\sum_{i=1}^{m} D(0,T_i + 2bd)v_i K$. Multiplying and dividing simultaneously by $(1+\tau_{i+1}F_{i+1}(T_i))$, the valuation of the floating leg is equal to:

$$FL = E\left[\sum_{i=1}^{n} \frac{D(0,T_i + 2bd)}{D(0,T_i)} \frac{D(0,T_i)}{1+\tau_{i+1}F_{i+1}(T_i)} \tau_{i+1}F_{i+1}(T_i) * (1+\tau_{i+1}F_{i+1}(T_i))\right] \quad (1.2)$$

$$= E\left[\sum_{i=1}^{n} \frac{D(0,T_i + 2bd)}{D(0,T_i)} D(0,T_{i+1})\tau_{i+1}F_{i+1}(T_i) * (1+\tau_{i+1}F_{i+1}(T_i))\right] \quad (1.3).$$

Denoting by E^{1+1} the expectation under Q^{1+1} the T_{i+1} forward measure, we obtain:

$$FL = \frac{D(0,T_i + 2bd)}{D(0,T_i)} * D(0,T_{i+1}) \sum_{i=1}^{n} E^{1+1}\left[\tau_{i+1}F_{i+1}(T_i) + (\tau_{i+1}F_{i+1}(T_i))^2\right] \quad (1.4)$$

But we know (see for instance the paper on Libor and market models) that $F_{i+1}(T_i)$ is a martingale under Q^{1+1} the T_{i+1} forward measure, and hence its dynamic can be represented as a driftless lognormal martingale:

$$dF_{i+1}(T_i) = \sigma_{i+1}(t) F_{i+1}(T_i) dW_t^{i+1} \quad (1.5),$$

where W_t^{i+1} represents a Brownian motion (also called Winener process, see Wiener process) under Q^{1+1}. This finally leads to

$$FL = \frac{D(0,T_i + 2bd)D(0,T_{i+1})}{D(0,T_i)} * \sum_{i=1}^{n} \tau_{i+1}F_{i+1}(0) + (\tau_{i+1}F_{i+1}(0))^2 \exp\left(\int_0^{T_i} \sigma^2{}_{i+1}(t)dt\right) \quad (1.6)$$

where the convexity adjustment $\exp\left(\int_0^{T_i} \sigma^2{}_{i+1}(t)dt\right)$ is directly inferred from corresponding caplet prices. In contrast to the plain vanilla swap, the in-arrear swap floating depends on the volatility of the forward rates through the caplet volatility[2]. As one would expect from the additive and one-rate per time nature of the in-arrear swap, this product does not depend on the correlations between the different rates.

The static replication pricing methodology diverges from the previous one when computing the expression (1.4). Basically, one needs to price the following expression $E^{1+1}\left[\tau_{i+1}F_{i+1}(T_i) + (\tau_{i+1}F_{i+1}(T_i))^2\right]$. This problem can be generalized to an even more general problem of evaluating $P = E^{1+1}[f(F_{i+1}(T_i))]$ where $f(.)$ is a general function, satisfying some regularity conditions (that can be ignored at first reading).

[2] hence the in-arrear swap can be considered to be a caplet product as opposed to the CMS swap that is more a swaption product (see CMS CMT swaps)

The trick is to realize that the market provides a very rich information via the caplets/floorlets prices. Indeed a caplet price with expiration (and reset) date T_i and payment date $T_{i+1}+2bd$, is given by

$$Caplet(T_{i+1},K)=D(0,T_{i+1}+2bd)E^{1+1}\left[(F_{i+1}(T_i)-K)^+\right] \tag{1.7}$$

where $(x)^+ = Max(x,0)$ and K is the caplet strike. Differentiating the above expression twice with respect to the strike leads to the fact that the density function of the forward libor $\varphi(F_{i+1}(T_i))|_K$ at the strike is given by the second order derivative of the caplet price with respect to the strike divided by the discount factor at time $T_{i+1}+2bd$:

$$\frac{\frac{\partial^2}{\partial K^2}Caplet(T_{i+1},K)}{D(0,T_{i+1}+2bd)} = \varphi(F_{i+1}(T_i))|_K \tag{1.8}$$

Using the density information found above, the pricing problem reduces to

$$P=\int_0^{+\infty} f(K)\frac{\frac{\partial^2}{\partial K^2}Caplet(T_{i+1},K)}{D(0,T_{i+1}+2bd)}dK \tag{1.9}$$

Integrating by parts twice leads finally to an expression of the type

$$P=\frac{1}{D(0,T_{i+1}+2bd)}\int_0^{+\infty}\frac{\partial^2}{\partial K^2}f(K)*Caplet(T_{i+1},K)dK \tag{1.10}$$

provided that $\left[f(K)*\frac{\partial}{\partial K}Caplet(T_{i+1},K)\right]_0^{+\infty}=0$, and $\left[\frac{\partial}{\partial K}f(K)*Caplet(T_{i+1},K)\right]_0^{+\infty}=0$ conditions that are satisfied in most cases. The expression (1.10) states the derivatives to price can be seen as an infinite sum of weighted caplets (whose weights are in fact equal to $\frac{1}{D(0,T_{i+1}+2bd)}\frac{\partial^2}{\partial K^2}f(K)$. Note that this approach is model free in the sense that it does not make any modelling asumptions and provides directly the hedging portfolio. It basically states that in-arrear swaps can be statically replicated by caplets or equivalently by a strip of caps.

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[3] The views and opinions expressed herein are the ones of the author's and do not necessarily reflect those of Goldman Sachs