Singular payoff options

INTRODUCTION

Singular payoff options refers to both contingent premium and binary options. This terminology comes from the fact that in both cases, the payoff shows a singularity jumping from zero to a non zero value. Because these options do not share much in common at first sight, we will review them separately. However, when pricing contingent premium option, we will see that these options can be decomposed into standard vanilla options and digital options.

DIGITAL OPTION

Also referred to as binary, cash-or-nothing or all-or-nothing options, digital option in its most vanilla version[1] pays a predetermined amount Q if the underlying asset is above (respectively below) a certain strike for a digital call (respectively put). Common exotic versions of the digital are:

- the American digital paying a certain amount if the trigger condition is activated during the life of the option
- the correlation digital paying something If another asset triggers the payment or non payment.

TARGET MARKET

[1] Called the simple European digital option

Digital are pure bet options. If the trader, investor is right, she will receives a known amount. The upside is even simpler than for standard call as one knows in advance what he should get. Because of this simplicity, digital options attract many market participants in the over-the-counter marketplace. Besides, structurers can use digitals to create a desired payoff profile. It is also worth noticing that some common structures are essentially series of digitals like for instance range accrual notes in fixed income market or mini-premium foreign exchange options.

Europen digital options are very easy to price under Black Scholes as their pricing consists in simply computing the probability for a stock of being above or below a given threshold. For instance for a binary call, the price is given by

$$BinC = e^{-rT} E\left[Q 1_{\{S_T > K\}}\right] = Qe^{-rT} P\left(S_T > K\right) = Qe^{-rT} N\left(d_2\right) \tag{1.1}$$

with $$d_2 = \frac{\ln\left(S_0 / K\right) + \left(r - q - \sigma^2 / 2\right)T}{\sigma\sqrt{T}} \tag{1.2}$$

where $N(x)$ is the cumulative normal density function, S_0 is the spot stock price, K the strike price, r the risk free rate, q the continuous yield dividend, T the option maturity and σ the Black Scholes implied volatility.

However, life is not as simple as the Black Scholes model assumes. Pricing with Black Scholes would be quite misleading for very skewed market. Binary options are well known to be very sensitive to the product of the slope of the smile (at the barrier level) and its vega. An intuitive explanation is to approximate the binary option by a call spread. Obviously, the payoff of a binary option is just the limiting case of a call spread:

$$1_{\{S_T > K\}} = \lim_{\varepsilon \to 0} \frac{(S_T - K + \varepsilon)^+ - (S_T - K - \varepsilon)^+}{2\varepsilon} \tag{1.3}$$

Therefore, writing that call option prices are function of the strike, the maturity, and the implied volatility itself function of the maturity and strike $C(T, K, \sigma(T, K))$, we have that the price of a binary should be equal to the limit of the call spread

$$BinC = \lim_{\varepsilon \to 0} \frac{C(T, K + \varepsilon, \sigma(T, K + \varepsilon)) - C(T, K + \varepsilon, \sigma(T, K - \varepsilon))}{2\varepsilon} \tag{1.4}$$

Or using the chain rule

$$BinC = \frac{\partial}{\partial K} C(T, K, \sigma(T, K)) + \frac{\partial}{\partial \sigma} C(T, K, \sigma(T, K)) \frac{\partial \sigma(T, K)}{\partial K} \tag{1.5}$$

Proper modelling of the volatility smile is therefore essential *(see Volatility skews and smiles).* However, for European digital option like this is the case, there is no need to use a complicated modelling of the volatility as one can use static replication to price the European option[2] (see *static replication* as well as *forward volatility contracts*).

AMERICAN DIGITAL AND CORRELATION DIGITAL

American digital option provides a predetermined amount Q if the strike price is touched at any time during the life of the option. Using standard result on the maximum of a Geometric Brownian motion, one can value these option easily under Black Scholes. For instance the price of an American digital option in a very general setting is given by:

[2] Based on the work of Breeden Litzenberger, one can find a portfolio of call and puts that create a synthetic digital options.

$$AmBinC = \left(\frac{K}{S_0}\right)^{q_1} \exp\left\{-\left(r + vq_1 - \sigma^2 q_1{}^2 / 2\right)\right\} T_s \left\{N_2\left(D_1, -DD_1, c\right) + N_2\left(-D_1, DD_1, c\right)\right\}$$

$$+\left(\frac{K}{S_0}\right)^{q_{-1}} \exp\left\{-\left(r + vq_{-1} - \sigma^2 q_{-1}{}^2 / 2\right)\right\} T_s \left\{N_2\left(D_{-1}, -DD_{-1}, c\right) + N_2\left(-D_{-1}, DD_{-1}, c\right)\right\}$$

(1.6)

where

$$D_v = \frac{\ln\left(S_0 / K\right) + \left(r - g - \sigma^2 / 2\right)\sqrt{T_s}}{\sigma\sqrt{T_s}} - \sigma\, q_v \sqrt{T_s} \tag{1.7}$$

$$DD_v = \frac{\ln\left(S_0 / K\right) + \left(r - g - \sigma^2 / 2\right)\sqrt{T - T_e}}{\sigma\sqrt{T - T_e}} - \sigma\, q_v \sqrt{T_s + Te} \tag{1.8}$$

$$c = -\sqrt{\frac{T_s}{T_s + T_e}} \tag{1.9}$$

$$q_v = \frac{v + v\sqrt{v^2 + 2r\sigma^2}}{\sigma^2} \tag{1.10}$$

for $v = 1$ or -1, with T_s , respectively T_e and respectively T the time when the barrier starts to be effective, respectively ends, respectively the maturity time of the option, $N_2(a, b, \rho)$ is a the cumulative function of a standard bivariate normal distribution with upper bound a and b and correlation of ρ .

One can notice that the computation is very similar to the one of barrier options. In fact, one can see digital option as a degenerated case of an American barrier. Again similar problem with the smile needs to be tackled. Last but not least the pricing of correlation digital requires a good knowledge of the correlation between the two assets and follows the same line in term of pricing as an outside barrier. Concepts of copula and cointegration can be useful to tackle the issue of meaningful correlation.

CONTINGENT PAYMENT OPTION

Option buyers do not like to pay option premium to wait and see that the option has expired out-of-the-money. The regret of paying up-front premium can be diminished with contingent payment options called pay later and pay-as-you go or money-back options depending on the type. When buying a pay later option, the option holder do not pay an upfront option premium. In contrast, at maturity, she pays a pre-agreed premium only if the option ends up in the money. Clearly, pay-latter or pay-as-you go options capture the investor’s desire to avoid unnecessary payment for option expiring worthless. However, this option is not without risk since the option can end up slightly in the money but with a payoff smaller than the premium to pay.

These options are easy to value under Black Scholes, as one requires computing the expected probability of ending in the money:

$$CO = e^{-rT}.E\left[\left\{w(S_T - K) - Q\right\}1\left\{wS_T > wK\right\}\right] \qquad (1.11),$$

where Q is the option premium, w equals 1 for a call -1 for a put and $1\{wS_T > wK\}$ is worth 1 if $wS_T > wK$ and 0 otherwise. The solution of (1.11) is given by

$$CO = wS_0 e^{-gT} N\left[wd_2 + w\sigma\sqrt{T}\right] - (wK + Q)e^{-rT} N(wd_2) \qquad (1.12),$$

with d_2 given by equation (1.2).

Like for digital, using Black Scholes model can be misleading for skew markets. And in fact, we can see that a contingent premium option can be decomposed into a standard vanilla options and a digital since

$$\{w(S_T - K) - Q\}1\{wS_T > wK\} = \{w(S_T - K)\}^+ - Q1\{wS_T > wK\} \tag{1.13}$$

Pricing and risk management of contingent premium option is therefore similar to the one of digital option. Static replication can be used for European version of contingent premium option while one needs correct modelling of the volatility smile when targeting American and Bermudan version of it. Similarly copula theory can bring valuable insight when targeting correlation contingent premium options.

Entry category: options.

Scope: Product description, risk management and pricing.

Related articles: contingent premium, binary option, exotic option.

Eric Benhamou[3]

Swaps Strategy, London, FICC,

Goldman Sachs International

[3] The views and opinions expressed herein are the ones of the author's and do not necessarily reflect those of Goldman Sachs

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