Leverage options

INTRODUCTION

Leverage options are options that even more leveraged than standard options. Although any option can be highly leveraged, leveraged options commonly refer to power option also called non-linear payoff options. Their lambda can be substantially higher than the corresponding vanilla option.

The name of power options comes simply from the power algebraic function used to compute the payoff. There are two types of power options:

- Symmetric power option (SPO): the power is both on the underlying and the strike with a payoff of the type:

  $$SPO = Max\left\{\left[w\left(S_T - K\right)\right]^p, 0\right\} \qquad (1.1),$$

  where w equals 1 for call option and –1 for put option, K is the strike price and S_T is the underlying price.

- Asymmetric power option (APO): the power is only on the underlying with a payoff of the type:

  $$APO = Max\left\{w\left(S_T{}^p - K\right), 0\right\} \qquad (1.2),$$

  with the same convention as above.

Obviously the power options degenerates to vanilla cases for a power of 1 ($p = 1$). And also, for power greater than one, power options are worth more than the corresponding European option, providing extra leverage.

TARGET MARKET

Power options are widely used in almost all markets: equity derivatives, foreign exchange, commodity and fixed income to provide high leverage strategy. For power equal to 2, the power option is called the parabola or squared contract[1], for 3 the cubic contract.

PRICING

There are various ways of pricing power options. Although the correct pricing should be using static replication, it is always instructive to derive closed form to get the nature of the pricing problem.

Black Scholes pricing

It is easy to compute the price of power options under Black Scholes assumptions:

- European Asymmetric option:

$$E^{Q_T}\left[e^{-rT}APO\right]=e^{-rT}\left\{wS_0{}^p \exp\left[p\left(r-g+\frac{1}{2}(p-1)\sigma^2\right)T\right]N\left(wd_p+wp\sigma\sqrt{T}\right)-wKN\left(wd_p\right)\right\} \quad (1.3)$$

where $N(x)$ is the cumulative normal density function, and S_0 is the spot stock price, K the strike price, r the risk free rate, q the continuous yield dividend, T the option maturity σ the Black Scholes implied volatility and

$$d_p=\frac{\ln(S_0/K)+\left(r-g-\sigma^2/2\right)\mathrm{T}+(1-1/p)\ln K}{\sigma\sqrt{T}} \quad (1.4).$$

[1] For instance option paying the Libor to the power 2 is called a Libor square contract.

- European symmetric options: these options are easy to value using the binomial expansion or the Taylor expansion method[2] that the price is given by:

$$E^{Q_T}\left[e^{-rT}SPO\right]=e^{-rT}w\left\{\sum_{i=0}^{M_p}(-1)^i\binom{p}{i}S_0^{\,p-i}K^i\exp\left[(p-i)vT+\frac{1}{2}(p-i)^2\sigma^2T\right]N\left(w\left[d+(p-i)\sigma\sqrt{T}\right]\right)\right\} \quad (1.5)$$

with the same notation as above and $\binom{p}{i}=\dfrac{\prod_{k=0}^{i-1}p-k}{i!}$

$$\begin{array}{ll} M_p=p & \text{if } p \text{ is an integer} \\ M_p=\infty & \text{otherwise} \end{array} \qquad d=\frac{\ln(S_0/K)+\left(r-q-\sigma^2/2\right)T}{\sigma\sqrt{T}} \quad (1.6)$$

However, the Black Scholes methodology, although providing nice closed formulae, can lead to serious mispricing on power option as higher order moments are mispriced. The flaw of the Black Scholes model comes from the non lognormality of return referred to as the smile effect.

One partial solution is to use mixture of lognormal to match the smile on various points but this still suffers partially from the smile effect. Other strategy using closed forms relies on using jump model like in Merton.

Complicated solution is to use a local volatility model. However, as explained below, there is a much easier and safer way of computing accurate power option prices using the distribution density and referred to as the static

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[2] To price these options, one needs first to deal with the easy case of integer power and use the binomial expansion: $(a-b)^p=\sum_{i=0}^{p}(-1)^i\binom{p}{i}a^{p-i}b^i$. For non integer power, one can use the Taylor

replication pricing methodology since it provides at the same time the static replicating portfolio.

Static Replication

The pricing problem of power option and more generally any European option with payoff $f(T, S_T)$ boils down to evaluate a forward neutral expectation given by:

$$E^{Q_T}[f(T, S_T)] \tag{1.7}$$

where in the case of a power option the function $f(T, S_T)$ is given by $f(T, S_T) = Max\{[w(S_T - K)]^p, 0\}$ for symmetric power options and by $f(T, S_T) = Max\{w(S_T{}^p - K), 0\}$ for asymmetric power options.

But using the result first derived by Breeden and Litzenberger (1978) and following Neuberger (1996), one know that the forward neutral density $\phi(S_T = K)$ is provided by the second order derivatives of the call option $C(S_0, K, T)$ (and also the put option) with respect to the strike:

$$\phi(S_T = K) = \frac{1}{B(0,T)} \frac{\partial^2}{\partial K^2} C(S_0, K, T) \tag{1.8}$$

Using this result, the payoff to integrate (equation 1.7) can be expressed in the form of a simple integral:

$$E^{Q_T}[f(T, S_T)] = \int_0^\infty f(T, K) \frac{\partial^2}{\partial^2 K} C(S_0, K) dK \tag{1.9}$$

which can be integrated by parts to provide the static replication hedge:

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expansion for $(1 - z)^p = \sum_{i=0}^{\infty} (-1)^i \binom{p}{i} z^i$ under the condition of $|z| < 1$

$$E^{Q_T}\left[f(T,S_T)\right]=\left[f(T,K)\frac{\partial}{\partial K}C(S_0,K)\right]_0^{\infty}-\left[\frac{\partial}{\partial K}f(T,K)C(S_0,K)\right]_0^{\infty}+\int_0^{\infty}\frac{\partial^2}{\partial K^2}f(T,K)C(S_0,K)dK$$

(1.10)

This shows interesting property of the static replication hedge:

- The terms in the bracket has to be finite for the static replication to make sense $\left[f(T,K)\frac{\partial}{\partial K}C(S_0,K)\right]_0^{\infty}-\left[\frac{\partial}{\partial K}f(T,K)C(S_0,K)\right]_0^{\infty}$. This implies in particular that $f(T,0)=0$ and $\frac{\partial}{\partial K}f(T,0)=0$ [3]. Also, one can show that a payoff function of the type exponential cannot be statically replicated when the underlying follows a lognormal diffusion.
- The static replicating hedge has weights equal to the second order derivative of the payoff function with respect to the underlying variable. This explains for instance that a parabolla contract is basically replicated with a portfolio of weight one of call option for all strikes. More generally a power option has weights of the form $p(p-1)\left[w(S_T-K)\right]^{p-2}1\{w(S_T-K)>0\}$ for symmetric case and $wp(p-1){S_T}^{p-2}1\left\{w{S_T}^{p}>wK\right\}$ for asymmetric one.

Another way of deriving the static hedging strategy is to use the replicating formula shown in Carr (1998). A Taylor Lagrange expansion up to order two gives that for any smooth function[4] $f(t,s)$ we have:

[3] These conditions are not very restrictive as one can always subtract to the payoff function a linear function of the form $f(T,0)+\frac{\partial}{\partial K}f(T,0)S_T$ to ensure that these two conditions are satisfied.

[4] by smooth, we mean almost surely twice differentiable and with continuous second order derivatives function.

$$f(T,S) = f(T,u_0) + \frac{\partial}{\partial K} f(T,u_0)(S-u_0) + \int_{u_0}^{S} \frac{\partial^2}{\partial K^2} f(T,u)(S-u)du \qquad (1.11)$$

which can be summarized into two simple cases using call and put payoff:

- if $S < u_0$ $\quad (S-u) = -(u-S)^+$ for any $S < u < u_0$ hence

$$f(T,S) = f(T,u_0) + \frac{\partial}{\partial K} f(T,u_0)(S-u_0) + \int_{S}^{u_0} \frac{\partial^2}{\partial K^2} f(T,u)(u-S)^+ du \qquad (1.12)$$

- if $S > u_0$ $\quad (S-u) = (S-u)^+$ for any $S > u > u_0$ hence

$$f(T,S) = f(T,u_0) + \frac{\partial}{\partial K} f(T,u_0)(S-u_0) + \int_{u_0}^{S} \frac{\partial^2}{\partial K^2} f(T,u)(S-u)^+ du \qquad (1.13)$$

Integrating the formulae above and taking into account discounting provides

the final replication formulae:

$$E^{Q_T}[f(T,S_T)] = f(T,u_0) + \frac{\partial}{\partial K} f(T,u_0)\left(E^{Q_T}[S] - u_0\right) + \int_{MinDf}^{u_0} \frac{\partial^2}{\partial K^2} f(T,u) \frac{1}{B(0,T)} Put(Strike = u)du$$
$$+ \int_{u_0}^{MaxDf} \frac{\partial^2}{\partial K^2} f(T,u) \frac{1}{B(0,T)} Call(Strike = u)du$$
$$(1.14)$$

where the variable u_0 can be choosen freely hence smartly and $MaxDf$ (respectively $MinDf$) is the upper (resp. lower) bound of the definiton domain of f with respect to its second variable. Obvious interesting choice are to take u_0 as

- the forward $u_0 = E^{Q_T}[S]$ since this will cancel the first order derivative term.
- to cancel the term $f(T,u_0) = 0$ (1.15)
- to cancel the term $\frac{\partial}{\partial K} f(T,u_0) = 0$ (1.16)

The two last choice can lead to multiple choice for u_0 that needs to solve the implicit equation (1.15) or (1.16).

There are a few practical consideration to know when using or static replication strategies:

- The market does not provide quote for all strikes. The static replication pricing requires a very broad range of option prices that are often not quoted accurately and not very liquid in the market. Traders' inputs for the extrapolation of the market are often necessary to feed the static replication pricer.
- The static replication assumes that the market quotes options for continuity of strikes. Real case includes a step between the available strike hence introducing ceteris paribus a discretisation error.
- The static hedge is not very feasible, as it requires many options and often-illiquid ones.

Partial solution is to provide certain liquid points and see how to get the best hedging strategy using only these options. A practical way of investigating this is to do a regression of a Monte Carlo based simulation using local volatility[5] model in order to price accurately the smile.

Entry category: options

Scope: power, curve linear option

Related articles: Lambda (option leverage).

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[5] with probably a jump to have a more realistic forward smile volatility surface as the jump component would force the forward smile to decrease slowly than in a pure local volatility model.

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[6] The views and opinions expressed herein are the ones of the author's and do not necessarily reflect those of Goldman Sachs

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