Dynamic replication

A key idea in modern finance is the replication of one portfolio of financial assets and/or liabilities by another. The existence of a replicating portfolio is directly linked to the assumption of market completeness. Combined with the absence of opportunity of arbitrage *(see arbitrage pricing and arbitrage theorem)*, the concept of replication leads to two important consequences:

- ability to price
- and ability to hedge.

First, the absence of opportunity of arbitrage implies that replicating portfolios should have the same price. Second, the existence of a replicating portfolio implies that one portfolio can be hedged by taking the offsetting position with the other portfolio. Often, the replicating portfolio is a dynamic portfolio in the sense that one needs to trade continuously to hold this replicating portfolio. Dynamic replication is often opposed to static replication that implies one-off trade.

Mathematically, concepts of replication involve the existence of a equivalent martingale measure *(see martingales)*, while the absence of opportunity of arbitrage implies the uniqueness of this martingale measure. A portfolio is said to replicate another portfolio if it generates at maturity identical cash flows (positive or negative).

Dynamic replication is the foundation of the Black-Scholes model *(see Black-Scholes-Merton option pricing model)*. In this model, under various conditions, like:

- no tax or transaction costs,
- ability to trade continuously the underlying asset and in particular to short-sell it,
- ability to borrow and lend at the risk free rate
- and constant volatility of the underlying security,

one can hedge the derivatives security by continuously trading a given quantity of the underlying security, referred to as the delta. Moreover, in this model, one can take advantage of mis-pricing by entering the delta hedging strategy and locking in a sense the arbitrage.

Other example of dynamic replication includes benchmark tracking by investment funds, pension and insurance fund surplus management and large-scale portfolio compression for fast VAR and other risk management evaluations.

Unfortunately, life is much more complex than the Black Scholes model and it is important to understand the concept of dynamic replication in a more general framework and its limitation. Strictly speaking, there exists no perfect dynamic replicating strategy but only attempts to dynamically hedge derivatives. Practitioners are forced to do much more than a simple delta hedging strategy. They need to hedge higher order terms (for instance gamma trading) as well as their risk with respect to model parameters (for instance vega trading).

They are also forced to take into account various artefacts like the presence of transaction costs, the violation of constant volatility and the constraint on ability to trade.

Transaction costs (*see trading (transaction) costs*), by itself, is already a wide subject. Under transaction cost, it becomes very costly to rebalance the hedge frequently. Attempts to model transactions costs have lead to more or less successful models like the Leland and Boyle-Vorst model (*see Leland (1985) model*). Moreover, under the presence of transaction costs, the concept of uniqueness of price does not hold any more. To come up with a price for a given strategy, various concepts have been developed:

- Super hedging strategy, as explained in Bensaid-Lesne-Pages-Scheinkman. A super-hedge is defined as a portfolio that will generate greater or equal cash-flows in any outcome. A super-hedge guarantee to make no loss as the super-hedge more than offsets the derivatives security. Drawbacks of this method are often the extra cost implied by the super-hedge strategy.
- Utility based approach as explained in Constantinides Davis-Zariphopolou. This approach breaks down the idea of a unique price as in the Black Scholes model since different agents will have different utility function and therefore different prices.
- Shortfall and risk measure. The idea is to find the cheapest strategy that combined with the derivatives security, leads to a given limited shortfall.

The other assumption of constant volatility is the milestone of derivative pricing. Many approaches have been developed such as stochastic volatility models (model of Heston, model of Hull and White) and its extension to uncertain volatility models, like in the Avellaneda-Levy-Paras model, econometric analysis of implied volatilities, mainly with PCA (Principal Component Analysis) and GARCH models, deterministic volatility model, like the ones of Dupire Derman and Kani , jump and Levy process models (Merton) as well as combination of the previous approaches (jump in the deterministic volatility models, as explained in Andersen and Andreasen) *(see Volatility skews and smiles)*

Although Breeden-Litzenberger showed that the knowledge of call prices for any strikes leads to the knowledge of the density function and consequently the Arrow Debreu prices, the dynamic of the underlying is crucial to the good pricing of path dependent exotic, and in particular for hedging barrier options and other exotics like the passport option (*see Passport (perfect trader) options*).

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[1] The views and opinions expressed herein are the ones of the author's and do not necessarily reflect those of Goldman Sachs

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