

Cash-Futures arbitrage processes

Cash futures arbitrage consisting in taking position between the cash and the futures markets to make an arbitrage. An arbitrage is a trade that gives in the future some free money with no risk (arbitrage are also referred by traders to as a free lunch). Because of the dual quotation of similar asset in the spot and futures markets, one can show that simple no-arbitrage relationship has to hold, otherwise one can make some free money. The relationship between cash and futures market is easy to derive. Hence, trading institutions have rapidly implemented popular program trading systems, fed by real time market quote, that consistently hunt for arbitrage opportunities between the two markets.

Futures markets are derivatives markets (see futures market overview) selling exchanged traded contracts that allow to buy or sell in some future date a given quantity of an asset. But the same asset can be also quoted in the spot market. Take for instance the case of crude oil contracts. Crude oil contracts exist both in the spot and futures market. And although a futures position may seem different from a spot position today's, it may not always be the case. For instance if on July 2002, one:

- Borrows some cash to buy 1,000 barrels of oil in the spot market (with the different quality: Dubai Brent, Brent Blend or WTI) and wait until the expiration date of the September futures contract,

□ or enters into a Futures contract (either on the IPE or the NYMEX) with delivery date September and pay every day up to maturity the margin calls,

one gets at maturity the same asset.

This means that there should be some relationship between cash or spot market and futures markets. The standard theory of market efficiency states that financial markets provide enough information for arbitrage to be spotted and eliminated quickly. This is often referred to as the golden rule of no-arbitrage condition.

One of the most standard arbitrage is precisely the no-arbitrage condition between spot and futures market and referred as the cash and carry strategy when buying the underlying and reverse cash and carry in the opposite situation. Forward contracts are easier to value and are in most cases giving same pricing results as futures contracts. For the ease of the presentation, we will first look at the relationship between spot and forward contracts before discussing the case of futures contracts, discussing how to compute the convexity adjustment.

Let us first examine the simple case of a forward contract on an asset with price dynamics (S_t) , paying no income during its total life. An investor can borrow some money today $t = 0$, buys the asset at the spot price S_0 and pays back the loan at the expiration T of the forward contract. Today, she pays no value. At expiry of the forward contract T , she is long the asset and

needs to pay back the loan, i.e.; she needs to pay $S_0 \exp\left(\int_0^T r_s ds\right)$, where r_s is the continuous risk free rate. What the investor has synthetically created is a contract that provides the asset at the expiration date T of the forward contract at a price of $S_0 \exp\left(\int_0^T r_s ds\right)$. This contract is similar in terms of its outcome to a standard forward contract. Therefore, if there exists no arbitrage, the forward contract price at expiry P_F should exactly be $S_0 \exp\left(\int_0^T r_s ds\right)$:

$$P_F = S_0 \exp\left(\int_0^T r_s ds\right) \quad (1.1)$$

If it was not the case, a clever investor would:

- sell the forward contract today $t = 0$ if the forward price P_F were

higher than $S_0 \exp\left(\int_0^T r_s ds\right)$. At the same time $t = 0$, she would buy

the asset at the price S_0 , financed by a loan. At expiry, he would deliver the asset to the forward contract holder, receive the forward price P_F and pay back the loan. The net outcome would be a

positive profit of $P_F - S_0 \exp\left(\int_0^T r_s ds\right)$. This arbitrage is known as cash

and carry as the investor uses some cash to immediately buy an asset in the spot market and carry it up to the maturity of the forward contract, while selling the equivalent forward contract.

- buy the forward contract today if the forward price P_F were lower

than $S_0 \exp\left(\int_0^T r_s ds\right)$. At the same time $t = 0$, he would short sell the

asset and put S_0 in the bank account. At expiry, he would receive the asset from the forward contract and lend it back. With the cash generated from the bank account, he would pay the forward contract and make the net profit of $S_0 \exp\left(\int_0^T r_s ds\right) - P_F$. As this arbitrage is exactly the opposite of the cash and carry arbitrage, it is referred to as the reverse cash and carry arbitrage

What we have obtained so far is that the forward contract price is equal to the spot asset value plus the cost of carry of this asset up to expiry.

$$\text{Fwd Price} = \text{Spot Price} + \text{Cost of Carry} \quad (1.2)$$

Example 1:

Consider a 3 month forward contract on a bond with current price \$910. We will assume that risk free rate (continuously compounded) is 3% per annum.

The following information leads the following forward price:

$$P_F = 910 \cdot \exp(3\% \cdot 3/12) = 916.85 \quad (E.1)$$

Short Selling

In fact the result above has made the implicit assumption that one can freely short sell an asset. In practice, short selling is not always permitted. However, even if it is the case, the no arbitrage relationship should still hold. This is because people holding the securities would still want to sell to make the reverse cash and carry arbitrage

Known income paying securities

Similarly to what we have seen for securities that do not provide any income like interest, dividend over the life of the forward contract, we can value securities that pay some known cash flows over the life of the contract.

When the cash flow is known, the same arbitrage (cash and carry or reverse and carry) can be made with the only difference that the investor will need to borrow from the bank a smaller amount of cash since the asset itself provides some additional known income. Because this income is known, the investor can plan in advance the additional revenue and reduce her/his cost of carry. Similarly, for commodity futures, the investor has to take into account the extra cost of storage (also called the **convenience yield**). As a consequence, the only difference lies in the **cost of carry** that is diminished by the known cash flows. All formulae for dividend paying and more are summarized in table 1.

Example 2:

Consider a seven month forward contract on a stock with current price \$55, discrete dividends of \$0.40 expected to be paid in three months, six months. Risk free rate is 3%. The present values of the different dividends is $0.40 \exp(-3\% \cdot 3/12) + 0.40 \exp(-3\% \cdot 6/12) = 0.79$ and the forward price is

$$P_F = (55 - 0.79) \exp(3\% \cdot 7/12) = 55.17 \quad (E.2)$$

General case for futures pricing

All the above examples were quite simple. In practice, futures and forwards contracts are slightly different because of the correlation between margin calls and interest rates. One has to compute an adjustment referred to as the convexity adjustment. Also the underlying of the futures contract may be denominated in another currency than the one of the futures, leading to a quanto correction. Table 1 provides an executive summary of the various possibilities when valuing a futures contract.

The notations used for the table are

- r is the rate used to compute the cost of financing, close to the risk free rate as read from the standard interest rate curve and adjusted by the funding cost of the trading desk
- q is the continuous yield dividend of the underlying asset of the futures contract
- g is the yield of the storage and transportation costs also called the convenience yield for commodity futures
- I is the present value of the different cash flows generated by the underlying asset of the futures contract
- $r^{foreign}$ is the foreign funding rate while $r^{dom} = r$ is the domestic one
- σ_{HL} is the yearly Ho&Lee volatility (typical values are around 1%)
- T is the time to maturity of the futures contract
- T_U is the time to maturity of the rate underlying the futures contract

- $\rho(FX, U)$ is the correlation between the FX rate X and the underlying asset U of the futures contract
- $\rho(IR, U)$ is the correlation between the spot interest rate IR and the underlying asset U of the futures contract
- X is the rate used to compute the quanto futures. The futures pays $S^{foreign}$ in domestic currency and X is domestic/foreign
- σ_x is the yearly volatility of the foreign rate
- $\sigma_{S^{foreign}}$ is the volatility of the foreign asset underlying the futures contract

We then have the following table to compute the fair value of the futures. Real life examples often include a combination of the simple case describe in the table 1. (see *Quantity-adjusting options (quantos)*)

Case	Value of the Futures
Cash Flow(s) with present value I	$(S_0 - I) e^{rT}$
Know continuous dividend yield q	$S_0 e^{(r-q)T}$
Storage cost g	$S_0 e^{(r+g)T}$
Foreign currency forward	$S_0 e^{(r^{dom} - r^{foreign})T}$
Convexity correction (this is added on top of the normal value of the forward)	$Exp\left(\frac{-1}{2} \sigma \rho (IR, U) T T_U\right)$
Quanto correction (this is added on top of the normal value of the forward)	$Exp(\rho(X, U) \sigma_X \sigma_U T)$

Table 1: Example of computation of Futures in order to compute the fair value of the basis

Key words: program trading, cash and carry arbitrage. Equity index futures, futures market, index arbitrage.

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¹ The views and opinions expressed herein are the ones of the author's and do not necessarily reflect those of Goldman Sachs

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