Brennan and Schwartz model (1982)

The Brennan and Schwartz model is a two factors model that models the dynamics of the short and long term rates. Simple one factor stochastic spot interest rates models cannot capture the multi-dimensionality of the yield curve term structure. To cut it short, one-factor models may account correctly for a parallel shift of the interest rates. But they would handle poorly the spread dynamics between short and long interest rates as well as the convexity of the curve.

For some instruments, this may be irrelevant, as they would mainly depend on one source of randomness. This is the case for vanilla interest rates options that can be efficiently priced in Black Scholes. However, for more sophisticated products, a two-factor model may be required.

Brennan and Schwartz (1982) introduced an early version of two stochastic factor models. Their model relies on the modelling of both the spot interest rates and the consol rates. The consol rates being a very long term yield includes in a sense information about the steepness of the curve.

Consol rates are defined as the yield of a consol bond. A consol bond is a perpetual bond that pays 1 every year for ever. If C_0 is the market value of a consol bond starting paying 1 in 1 year and every consequent year, its yield has to satisfy:

$$C_0 = \sum_{n=1}^{+\infty} \frac{1}{(1+l)^n} = \frac{1}{l} \qquad (1.1).$$

Generally, two factor modelling assumes that the two key factor dynamics can be written in terms of two correlated Brownian motions W_t^1, W_t^2 (see Wiener process) with a correlation of ρ. The dynamics of the factors is given by:

$$dr_t = \mu_r dt + \sigma_r dW_t^1, \quad (1.2),$$

$$dl_t = \mu_l dt + \sigma_l dW_t^2 \quad (1.3),$$

where are two. $\mu_r, \quad \mu_l, \quad \sigma_r, \quad \sigma_l, \quad \rho$ can be functions of $t, \quad r_t, \quad l_t$.

The approach of Brennan Schwartz assumes that the first factor is the spot rate while the second corresponds to the consol rate. Taking the consol rate as the second factor simplifies the problem as one needs to derive the no-arbitrage conditions relating the spot and the consol rates.

Since a consol bond is a tradable, denoting by $\lambda_r, \quad \lambda_l$ the market prices of risk for $r, \quad l$, the consol bond price C_0 has to satisfy the non-arbitrage PDE:

$$\frac{\partial C_0}{\partial t} + (\mu_r - \lambda_r \sigma_r)\frac{\partial C_0}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 C_0}{\partial r^2} + (\mu_l - \lambda_l \sigma_l)\frac{\partial C_0}{\partial l} + \frac{1}{2}\sigma_l^2 \frac{\partial^2 C_0}{\partial l^2} + \rho \sigma_r \sigma_l \frac{\partial^2 C_0}{\partial l \partial r} = rC_0 \quad (1.4)$$

Substituting the relation (1.1) into (1.4) leads to:

$$\mu_l - \lambda_l \sigma_l = l^2 - rl + \frac{\sigma_l^2}{l^2} \quad (1.5)$$

In the Brennan Schwartz model, the dynamics of the spot rate and consol rates are given by

$$dr = (a_1 + b_1(l - r))dt + \sigma_1 r dW_t^1, \quad (1.6),$$

$$dl = l(a_2 - b_2 r + c_2 l)dt + \sigma_2 l dW_t^2 \quad (1.7).$$

Because of the relatively complicated functional forms, there does not exist a simple solution for the pricing of zero coupon bonds. Although the stochastic part of the two diffusions is of lognormal type, the drift terms with reciprocal mean reversion and non linearity of the second diffusion hinder any easy analytical solving.

The model calibration is done by statistical estimation. Moreover, in the original paper the diffusions are under the historical measure, hence requiring some estimation of the market prices of risk. The key point of this model is to have two factors easy to translate in financial terms.

Brennan and Schwartz applies their model to analyze government bonds of the Canadian market. They show that this model can five satisfactory results in terms of pricing using a finite difference method to solve for the price.

However, this model has serious flaws out of which two have been strongly advocated:

1. The model does not have easy solution for simple instrument. The implied calibration is therefore very delicate as oppposed to affine model for the term structure.
2. The model can blow up in finite time, meaning that rates can go to infinity as shown by Hogan (1993).

Additionally, it is not easy to check the Heath Jarrow condition on this model. In response to the bad property of this model, a wide range of two factors models have been developed:

1. General affine models for the spot and consol rates. This leads to have a general solution for the zero coupon bonds of the form: $\exp(A(t,T) - B(t,T)r - C(t,T)l)$. Famous affine model is for instance the Hull and White two factor model (1994) (also referred to as a linear markovian Gaussian model).

$$\begin{aligned} dr &= (\nu(t) - u - \gamma r)dt + c dW_t^1 \\ du &= -au dt + b dW_t^2 \end{aligned} \tag{1.8}$$

   Extension of this class also includes the models of humped volatility term structure like the model of Moraleda, Mercurio and Moraleda, as well as general affine models Duffie and Kan

2. Square root diffusions like the fong and Vasicek (1992) model and Cox Ingersoll Ross ()multifactor models:

$$\begin{aligned} dx &= a(\bar{x} - x)dt + \sqrt{x} dW_t^1 \\ dy &= b(\bar{y} - y)dt + \sqrt{y} dW_t^2 \\ r &= cx + dy \end{aligned} \tag{1.9}$$

3. Formulation of the above models following the Heath Jarrow (1993) framework: like the two factor Hull and White model and many more.

4. Market models also referred to as Brace Gatarek models

Entry category: mathematical models

Scope:

Related articles: yield curve modelling.

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[1] The views and opinions expressed herein are the ones of the author's and do not necessarily reflect those of Goldman Sachs

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