Boundary conditions for options

Boundary conditions for options can refer

- to the non-arbitrage conditions that option prices has to satisfy. If these conditions are broken, arbitrage can exist.
- to the boundary conditions in a finite difference method and more generally any numerical methods like finite elements.

NON ARBITRAGE CONDITIONS

When one looks at the various non-arbitrage conditions that an option has to satisfy, one may think about:

- Rule 1: Options cannot have a negative price as the option holder would never exercise an option that provides her negative cash flows (except for very specific and rare reason like tax and or accounting issues).
- Rule 2: American options cannot be less expensive than the corresponding European option. American and European option can be exactly the same in certain cases, like the school case of the American call on stock paying no dividend.
- Rule 3: Option prices have to be higher than the discounted intrinsic value. In other words, the time value cannot be negative. Note that this rule implies in particular the first one. In particular the price of a butterfly whose payoff is always positive cannot be negative.

- Rule 4: Vanilla European option prices have to satisfy synthetic replication relationship like put call parity. American options only satisfy a weaker version of it.
- Rule 5: Vanilla options have to satisfy some basic monotonicity relationships like a call should decrease with higher strike. These relationships are summarised in table 1.
- Rule 6: Vanilla options has to satisfy certain boundaries in terms of admissible models. If no risk neutral pricing model can give the option price, there exists an arbitrage opportunity. In particular, if one cannot bootstrap for instance local volatility surface from market option prices, one may think that there exists some arbitrage opportunities.
- Rule 7: If two strategies are such that the payoff of strategy 1 always exceed strategy 2, then strategy 1's price has to be higher than strategy 2. This is the fundamental idea of super replication and leads for instance to the fact that a call option cannot be worth more than the stock itself.

All these rules are very useful when designing robust regression tests of an option pricing trading platform. Quantitative developers have to think these regression tests as sanity checks ensuring that none of these rules are broken as this would introduce arbitrageable prices in the system. In addition, sanity checks should include extreme cases like very high volatility, very low volatility, zero parameters and negative parameters to test the robustness of the system.

Factor	Call	Put
Spot price	↑	↓
Strike	↓	↑
Time decay or time to expiration	↑	↑
Risk free rate	↑	↓
Dividend and cost of financing	↓	↑
Implied volatility	↑	↑

Table 1: Factor affecting call and puts

BOUNDARY CONDITIONS FOR FINITE DIFFERENCE

Boundary conditions for finite difference and tree methods refer to the limit conditions used at the hedge of the numerical grid or mesh or tree. Broadly speaking, one often uses conditions of the following type:

- Dirichlet conditions: these conditions specify the value of the function at the boundary of the PDE. For instance for a standard option, this would say that the option is worth the intrinsic value at the boundary.
- Van Neuman conditions: these conditions specify the value of the first order derivatives function at the boundary of the PDE. For instance for a call option, this would say that the delta of the option is worth 0 for low strike and 1 for very high strikes.
- Second order derivatives conditions: these conditions specify the value of the second order derivatives function at the boundary of the PDE and more generally provides a linear relationship between the second and first

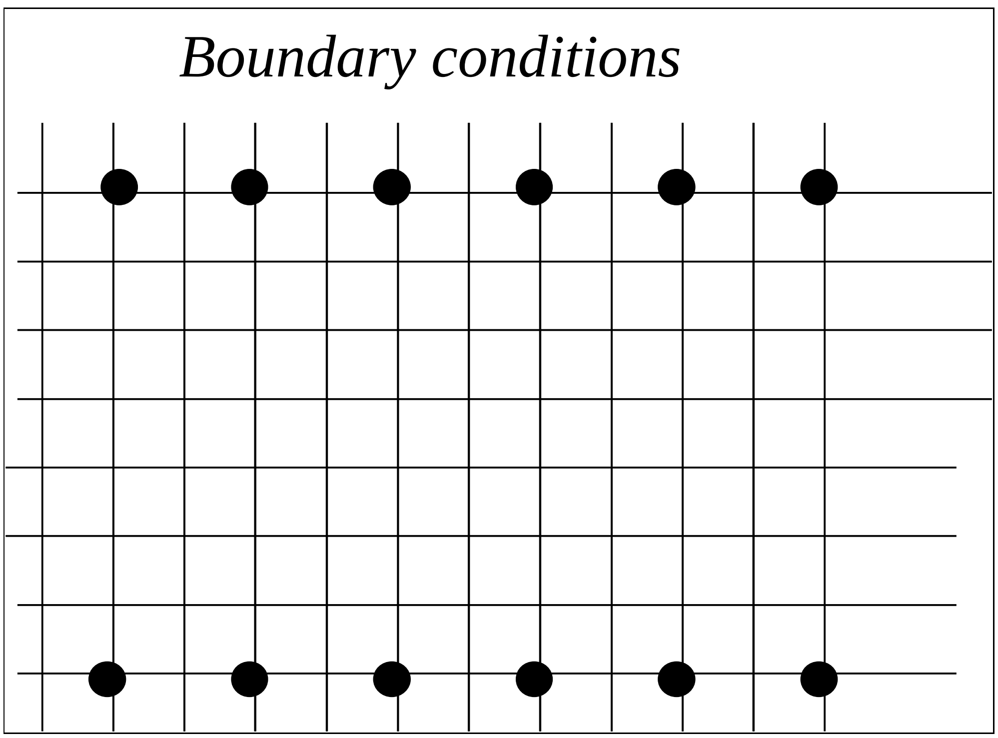
order derivatives function and the function itself. For instance if one says that the second order derivatives is null for large value of the underlying. $\frac{\partial^2 V}{\partial S^2}(S,t) \to 0$ as $S \to \infty$, one can represent this simply using the finite difference approximation: $V_I{}^k = 2V_{I-1}{}^k - V_{I-2}{}^k$

A useful boundary condition may be found when solving for the deterministic part of the PDE. In the case of a boundary with the state value equal to zero, the PDE becomes a first order linear ordinary equation of the type: $\frac{\partial V}{\partial t} - rV = 0$, which leads to a trivial discretized boundary of the form $V_o{}^k = (1 - r\delta t)V_0{}^{k-1}$.

In general, option are not very sensitive to the boundary condition except for barrier type option with the boundary very close to the barrier level. In this case, it is very important to take a boundary condition well suited for the problem as this option problem is extremely sensitive to the precision of the boundary conditions. For instance, in the case of an up and out call, if the boundary is not exactly at the barrier level and slightly above (in the case of a moving barrier that may not always be coinciding with the grid points), it is very important not to set the boundary point to zero for the nearest point from the barrier but to introduce a fictitious point in order to perturbe less the solution.

In general boundary condition do not influence the stability of the numerical scheme but rather its efficiency and speed to converge.

When designing a general grid model (general theta scheme), one should be careful in allowing at least for the three types of boundary conditions: Dirichlet Van Neuman and second order derivatives. This condition enables to fulfil the missing points in the corner of the diffusion matrix.



Picture 1: geometrical representation of a finite difference grid

Entry category: options

Scope: min max prices, value before expiration, American style vs European style, price test, min and maximum vol.

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